

Emergent bipartiteness in a society of knights and knaves

C I Del Genio and T Gross

Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, 01187 Dresden, Germany

E-mail: paraw@pks.mpg.de

Abstract. We propose a simple model of a social network based on so-called knights-and-knaves puzzles. The model describes the formation of networks between two classes of agents where links are formed by agents introducing their neighbors to others of their own class. We show that if the proportion of knights and knaves is within a certain range, the network self-organizes to a perfectly bipartite state. However, if the excess of one of the two classes is greater than a threshold value, bipartiteness is not observed. We offer a detailed theoretical analysis for the behaviour of the model, investigate its behavior in the thermodynamic limit, and argue that it provides a simple example of a topology-driven model whose behaviour is strongly reminiscent of a first-order phase transitions far from equilibrium.

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1. Introduction

“The only way out of here is to try one of these doors. One of them leads to the castle at the centre of the Labyrinth, and the other one leads to certain death! You can only ask one of us, and I should warn you that one of us always tells the truth, and the other always lies.” [1] The statement above poses a Knights-and-Knaves puzzle – a class of logic puzzles made popular by Raymond Smullyan [2]. As their defining feature, these puzzles contain two types of characters: the knights, who always tell the truth, and the knaves, who always lie.

In physics and mathematics, the investigation of simple puzzles and toy models has often led to deep insights. For instance, puzzles of the Knights-and-Knaves type are quoted as an inspiration for Gödel’s incompleteness theorem [3]. Examples of influential simple models from physics include the Ising model [4] and the Bak-Tang-Wiesenfeld model of self-organized criticality [5]. In the physics of complex networks, simple models have significantly advanced our understanding of both the topological evolution of networks [6, 7] and the dynamical processes taking place on them [8, 9]. More recently, studies of the adaptive voter model [10, 11], a highly simplified model of opinion dynamics, have resulted in a better understanding of adaptive networks, which are networks in which the topology coevolves with the state of the nodes [12].

Here, we propose a very simple network formation game [13, 14], inspired by Knights and Knaves puzzles. In contrast to traditional puzzles, the model (described in section 2) considers the dynamics of a social network of knights and knaves. We assume that every agent, regardless of his own character, tries to connect to knights while avoiding knaves. However, by nature of this game, every agent will claim to be a knight if asked directly. Therefore, the agents have to rely on social information, asking their neighbours with whom to link and whom to avoid.

One of the solutions to the puzzle posed above is to ask one of the agents which door the other agent would recommend. The agent will then invariably name the door that leads to death, thus implicitly revealing the door that leads to the castle. This solution exploits a symmetry of the puzzle: a knight relating the answer of a knave will result in the same information as a knave relating the answer of a knight. Alternatively, one can ask one of agents what he would recommend if asked directly. A knight will truthfully relate his true answer, while a knave will lie about his lie; in both cases the right door is named.

The symmetry of the knights-and-knaves puzzles carries over to the proposed network model. A knave will always recommend linking to knaves, pretending them to be knights. By contrast, a knight will always recommend linking to knights, truthfully revealing their knightly character. Thus, a symmetric situation arises in which every agent recommends those of his own type.

Based on the above, one might argue that the model has some significance for opinion formation processes, with, e.g., Republicans referring their discussion partners to other Republicans and Democrats referring to other Democrats. However, our main motivation for studying the network of knights and knaves stems from a different source: The model proposed here is one of the simplest nonlocal systems exhibiting nontrivial topological dynamics. Thus, it constitutes a step toward the exploration of mesoscale dynamics in networks.

In this paper we study the dynamics of the Knights-and-Knaves network numerically and analytically. One question that immediately comes to mind is whether the knights manage to separate themselves from the knaves. In a wide range of

parameters, the opposite turns out to be true: The network approaches a completely bipartite state in which every knight is connected only to knaves and every knave is connected only to knights. Bipartiteness is still achieved if there is a significant difference in the numbers of knights and knaves, but disappears when the difference exceeds a certain threshold. Our analysis reveals a strong analogy between the behaviour of the system and thermodynamic properties close to first order phase transitions. The proposed model may thus offer an analytically tractable example of such a transition in a topology-driven finite-temperature system far from equilibrium.

2. The model

We consider a network of T knights (T for truthful) and L knaves (L for liar), such that the total number of nodes is $N = T + L$ and the proportion of knights in the population is $f_T = T/N$.

The network starts from some random initial configuration and then evolves according to the following rules: In every time step we randomly chose a node, i , one of its neighbors, j , and one of j 's neighbors, $k \neq i$. If node j and k are of identical type (both T or both L) then i connects to k or maintains the connection to k if one exists already. If node j and k are of different type (one T, one L) then i does not connect to k and cuts the connection to k if one exists already. This procedure is iterated until the system reaches either an absorbing state, where no further change of the topology is possible, or a thermodynamic steady state, in which the microscopic dynamics continues.

For the analysis below it is useful to define $[TT]$ as the total number of links between knights, $[LL]$ as the total number of links between knaves, and $[TL]$ as the total number of links between a knight and a knave.

3. Numerical results

For investigating the phenomenology of the model we start by defining thermodynamic observables. Because we motivated the model by assuming that the agents aim to connect to knights, it is reasonable to introduce observables that measure how well this goal is achieved. We measure the success of knights by a parameter $p_T = 2[TT] / (2[TT] + [TL])$, denoting the proportion of neighbors of knights that are knights. Analogously, we define $p_L = [TL] / ([TL] + 2[LL])$ as the proportion of neighbors of knaves that are knights. Finally, we denote the probability that a knight is reached by following a random link as $p_A = (2[TT] + [TL]) / [2([TT] + [TL] + [LL])]$.

In simulations, we observe that if the number of knights equals the number of knaves then the system always evolves to a state where all neighbors of knights are knaves and all neighbors of knaves are knights, i.e., $p_T = 0$, $p_L = 1$ and $p_A = 0.5$. In the following we denote this state as the bipartite state of the network.

Once in the bipartite state the dynamics freezes: Given two nodes i and k , with common neighbour j , either i and k are knights while j is a knave, or i and k are knaves while j is a knight. Either way, j is of a type different from both i and k . Hence, no link between i and k can be placed and no link from i to k can exist, which could be removed.

We now ask whether the bipartite state can still be reached if the proportion of knights and knaves in the population is different. Simulations of the network dynamics for different values of N and f_T show that the equilibrium network is

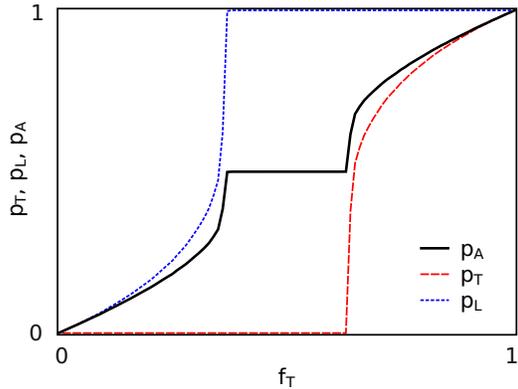


Figure 1. Emergent bipartiteness. Ensemble averaged probabilities for a neighbour of a knight to be a knight (p_T , dashed red line), for a neighbour of a knave to be a knight (p_L , dotted blue line), and for a neighbour of any node to be a knight (p_A , solid black line), as a function of the fraction f_T of knights in the network. The lines are averages over an ensemble of 10^4 networks with $N = 10^4$ nodes. The networks exhibit perfect bipartiteness in the region roughly between $f_T = 0.37$ and $f_T = 0.63$.

bipartite throughout a range of values of f_T centered around 0.5 (figure 1), whereas bipartiteness is lost if the proportion of knights or knaves exceeds some threshold. Thus, depending on the value of f_T three different regimes are observed: First, at low f_T the knights are exclusively connected to knaves, whereas knaves have additional connections among themselves. Second, at f_T around 0.5 both knights and knaves are exclusively connected to agents of the other type (bipartite state). Third, at high f_T the knaves are exclusively connected to knights, whereas the knights also have connections among themselves.

Let us emphasize that the behaviour of p_A is strongly reminiscent of the Maxwell construction for the isotherms of the van der Waals equation, or the M-H isotherms of magnetic systems undergoing first-order phase transitions [15]. In this context of phase transitions, the model should be considered as a systems at non-zero temperature. While we prescribed the result of the update of given triplet deterministically, the triplets to update are chosen randomly. This stochasticity constitutes a finite temperature, which is higher in networks of smaller size. We note that phase transitions can only be strictly defined for systems of infinite size. However, the space of possible topologies of networks scales with 2^{N^2} , where N is the number of network nodes. Therefore, even relatively small networks constitute a large configurational space, meriting the application of statistical concepts.

In figure 2, the value of p_A in the final state of the network is shown for different network sizes. We observe that the range of bipartiteness decreases with decreasing system size. If the network size is shrunk down to $N = 100$ nodes then the range of bipartiteness becomes a single point reminiscent of the critical point of the van der Waals equation. While this analogy should certainly be explored in more detail by artificially introducing noise in larger networks, this investigation is beyond the scope of the current paper. Instead, we focus on the dynamical origin of the bipartite regime.

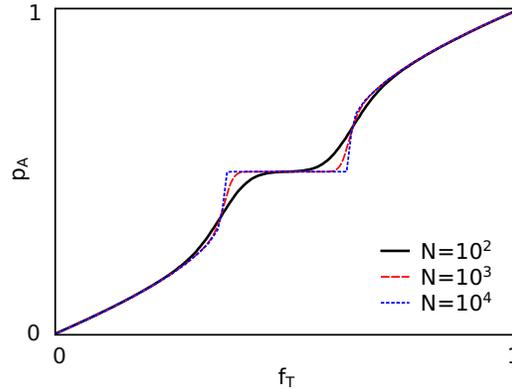


Figure 2. Effect of system size. Shown is the ensemble averaged probability for the neighbour of a node to be a knight (p_A), as a function of the fraction f_T of knights in the network, for different system sizes. The solid black line corresponds to $N = 10^2$, the dashed red line to $N = 10^3$, and the dotted blue line to $N = 10^4$. The bipartiteness range decreases with system size, and vanishes for $N = 10^2$. The lines for $N = 10^3$ and $N = 10^4$ are averages over ensembles of 10^4 networks; the line for $N = 10^2$ is an average over an ensemble of 10^6 networks.

4. Thermodynamic theory

The regimes described in the previous section are “thermodynamic”, in the sense that they present very probable, but not strictly certain, outcomes of the network evolution. To see this consider for instance a network containing only one knight among a large number of knaves, corresponding to a value of f_T well below the observed bipartite range. Even in this case it is still possible to construct a bipartite network configuration in which the knight is connected to every other agent and no further links exist in the system, such that the network is in an absorbing bipartite state. However, the probability that this configuration arises in the evolution of the network (in finite time) is so low that it is never observed in any network with more than a few nodes.

While in very small systems any of the three macro-states (bipartite network – connections between knights but not knaves – connections between knaves but not knights) could be observed with some probability, we observe that larger networks reliably select one of the three behaviors depending the control parameter f_T .

For understanding the mechanism behind this selection it is instructive to consider the distribution of *degrees* (i.e., the number of connections) for the agents of the two types. The results presented above show that, within a certain range, the probability that the neighbor of a random agent is a knight is 0.5 regardless of the precise density of knights in the network. This is possible because the agents of the type numerically in excess have a proportionally lower number of network connections per agent (see figure 3). However, if the proportion of knights or knaves becomes too small, the agents in the majority start forming connections among each other. The transitions bordering the bipartite regime can thus be understood as the nucleation of “droplets” of connected majority nodes from the bipartite mixture.

In the remainder of this paper we investigate the formation and breakdown of the bipartite state. Let us first motivate the existence of this state by a thermodynamic

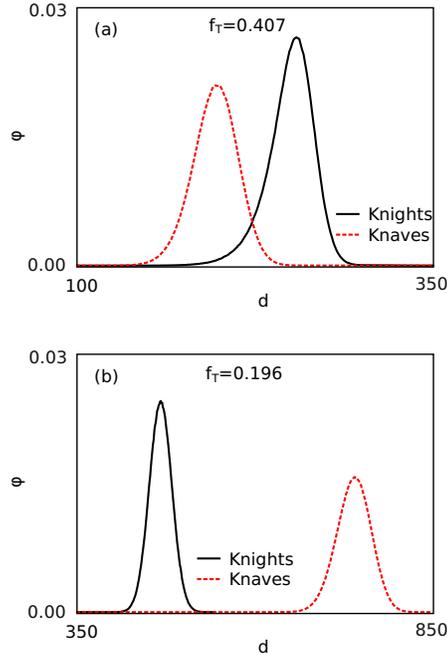


Figure 3. Ensemble averaged degree distributions φ for knights (solid black line) and knaves (dotted red line), with $N = 10^3$. The lines are averages over an ensemble of 10^4 networks. Panel (a) shows data for $f_T = 0.407$: the distribution of the majority nodes (knaves) “shifts” towards lower degrees, while that of the minority nodes (knights) shifts towards higher degrees. Panel (b) shows data for $f_T = 0.196$, outside the bipartiteness range: the minority nodes (knights) have consistently lower degrees than the majority ones (knaves).

argument, in which, for the moment, we forget our knowledge of the microscopic dynamics. It is clear that the mean degree of agents is strongly controlled by the microscopic rules and thus cannot be inferred from a purely thermodynamic perspective. This situation is similar to a thermodynamic system whose energy is controlled by coupling to an external heat bath. Therefore, we consider an ensemble of systems with a given mean degree, which is somewhat analogous to the micro-canonical ensemble of equilibrium statistical physics.

In the present non-equilibrium system there is no reason to believe that the microstates in the ensemble should be equiprobable. We nevertheless draw on the micro-canonical picture and adopt equiprobability as an admittedly naive working hypothesis. Under this hypothesis one can then argue that one should observe the macro-state corresponding to the largest number of micro-states. In other words, we expect to observe the bipartite state when the number of micro-states that are bipartite is greater than the number of micro-states in which connections between nodes of the same type exist.

In a network model a microstate corresponds to a distinct realization of the network topology, e.g. the precise pattern of neighborhood relationships. We could now proceed by computing closed expressions for the respective numbers of micro-states and asking at which value of f_T the micro-states corresponding to bipartiteness

are in the majority. While we will indeed derive similar expressions below, let us first follow a simpler approach leading to the same result. We estimate the relative number of configurations by considering the bipartite state and comparing the number of bipartite and non-bipartite configurations that are reached by rewiring one link.

Without loss of generality we assume that the knights are in the majority. In this case placing a link between two knights clearly leads to a larger number configurations than placing a link between two knaves. Given that [TL] bipartite links already exist in the network, the number of possibilities for placing the rewired link between a knight and a knave is

$$Q_{\text{TL}} = TL - [\text{TL}] ,$$

whereas the number of possibilities for placing a link between two knights is

$$Q_{\text{TT}} = \frac{T(T-1)}{2} [\text{TT}] .$$

The number of admissible bipartite configurations exceeds the number of admissible non-bipartite configurations when

$$Q_{\text{TL}} > Q_{\text{TT}} , \quad (1)$$

or, equivalently,

$$T < \frac{2}{3} (N - \langle k \rangle_{\text{T}}) , \quad (2)$$

where $\langle k \rangle_{\text{T}} = [\text{TL}]/T$ is the mean degree of the knights.

Following the reasoning above, we would expect to observe the bipartite state whenever the condition in Eq. (2) is met. The simple thermodynamic reasoning therefore predicts the observation of the bipartite state if the difference in the proportion of minority and majority nodes is sufficiently small. For gaining a quantitative estimate of the transition point we assume $\langle k \rangle_{\text{T}} \approx N/7$, which we observed in network simulations, independently of N . This yields a bipartite range of $0.43 \lesssim f_{\text{T}} \lesssim 0.57$.

The results in Fig. figure 4 show that the thermodynamic estimate of the transition point is of the right order of magnitude but differs by some percent from the value observed in network simulations. The discrepancy can be attributed to the simplicity of the thermodynamic estimation, which used the unwarranted assumption of equiprobability of states and neglected the microscopic dynamics. For obtaining a more precise estimate we formulate a microscopic kinetic theory in the following section.

5. Kinetic theory

In this section we use a network moment expansion [16, 17] to formulate a set of coarse-grained equations that capture the emergent-level dynamics of the system. Considering the effect of link creation and destruction processes on the abundance of links of a given type leads to the equations

$$\frac{d}{dt} [\text{TT}] = 2[\text{TTT}] - 2[\text{TLT}]_{\Delta} \quad (3)$$

$$\frac{d}{dt} [\text{TL}] = [\text{TLL}] + [\text{TTL}] - [\text{TLL}]_{\Delta} - [\text{TTL}]_{\Delta} \quad (4)$$

$$\frac{d}{dt} [\text{LL}] = 2[\text{LLL}] - 2[\text{LTL}]_{\Delta} . \quad (5)$$

where we used three-letter symbols indicate open triplets of nodes and triangles. For example, [TTT] indicates the number of open-chain triplets made of three knights, while $[\text{TLT}]_{\Delta}$ refers to the number of triangles composed of two knights and one knave. The symbols [TLL], [TTL], $[\text{TLL}]_{\Delta}$, $[\text{TTL}]_{\Delta}$, [LLL], [LTT], and $[\text{LTL}]_{\Delta}$ are defined analogously.

Because of the appearance of three-node motifs, the equations above do not constitute a closed system. We close the system by the so-called moment closure approximation [16, 18], which replaces the abundances of three node motifs by a statistical estimate based on the abundances of smaller motifs, e.g.

$$[\text{TTT}] = \frac{2[\text{TT}]^2}{T} - \frac{4[\text{TT}]^3}{T^3}, \quad (6)$$

$$[\text{TTT}]_{\Delta} = \frac{2[\text{TT}]^3}{T^3} \quad (7)$$

$$[\text{TLL}] = \frac{3[\text{TL}][\text{LL}]}{L} - \frac{4[\text{TL}]^2[\text{LL}]}{TL^2}, \quad (8)$$

$$[\text{TLL}]_{\Delta} = \frac{4[\text{TL}]^2[\text{LL}]}{TL^2}, \quad (9)$$

where we used the observation that the degree distribution is sufficiently narrow to be treated as poissonian and assumed the absence of correlations beyond the next neighbor. The remaining expressions are easily obtained from (6), (7), (8) and (9) exploiting the symmetry of the system, the invariance of triangles under permutation of their nodes, and the invariance of open triplets under exchange of their end nodes.

Applying the moment closure approximation to (3), (4) and (5), we obtain

$$\begin{aligned} \frac{d}{dt}[\text{TT}] &= \frac{4[\text{TT}]}{T} \left([\text{TT}] - \frac{2[\text{TT}]^2}{T^2} - \frac{2[\text{TL}]^2}{TL} \right) \\ \frac{d}{dt}[\text{TL}] &= [\text{TL}] \left(\frac{[\text{TT}]}{T} + \frac{[\text{LL}]}{L} \right) \left(3 - \frac{8[\text{TL}]}{TL} \right) \\ \frac{d}{dt}[\text{LL}] &= \frac{4[\text{LL}]}{L} \left([\text{LL}] - \frac{2[\text{LL}]^2}{L^2} - \frac{2[\text{TL}]^2}{TL} \right). \end{aligned} \quad (10)$$

The equation system, Eqs. (10), is analytically tractable. We analyze the system by computation of stationary states and a subsequent linear stability and bifurcation analysis. The bipartite state $[\text{TT}] = [\text{LL}] = 0$ is trivially stationary. In the range where bipartiteness is actually observed, this state is the only attractor of the system. If f_T is increased or decreased beyond the bipartite regime, a qualitative transition of the dynamics is encountered where two additional stationary states are formed (figure 4), one of which is dynamically stable. We can identify these transitions as saddle-node bifurcations (also called fold bifurcations in the mathematical literature).

The results in figure 4 show that the saddle-node bifurcations coincide very well with the transition points observed in the network simulations. We observe a small discrepancy between the estimated and observed quantities close to the border of the bipartite regime. Notably, the transitions bordering the bipartite regime look continuous in the network simulations but are discontinuous in the analytical approximation. These differences arise most probably because of finite size effects in

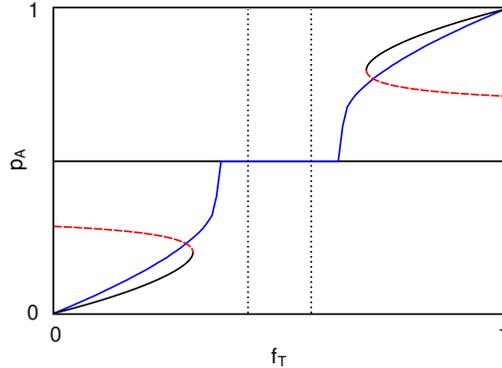


Figure 4. Comparison of the transition points. The figure shows the stationary probabilities for neighbours of any node to be knights (p_A), estimated via moment closure approximation (black and red lines) and by direct network simulation with $N = 10^4$ (blue line). Furthermore, transition points obtained from a simple “thermodynamic” estimate are shown (dashed grey lines). The analytical moment-closure approximation offers a very precise estimate of the transition points. In the approximation these points correspond to saddle-node bifurcations, where a stable (black) and an unstable (red) steady state collide and annihilate.

the network simulation or because of the presence of long-ranged correlations which are neglected in the moment-closure approximation.

Our main conclusion from the thermodynamic plausibility argument and the analytical model is that the qualitatively different regimes persist in the thermodynamic limit of infinite network size. We perceive this observation as a strong encouragement for considering the observed phenomenon as a phase transition.

6. State selection

The analytical approximation showed that within the bipartite regime, the bipartite state is the only attractor of the system. However, outside the bipartite regime a stable steady state coexists with the bipartite absorbing state. While network simulation showed that the system always approaches the non-bipartite state in this case, the same information cannot be obtained analytically from the differential equations alone. We explore this point further by combining results from the analytical approximation with the thermodynamic reasoning used above.

A drawback of our thermodynamic arguments was that we had to use heuristic values for the density of links between given types of agents. This drawback can now be mitigated by using results from the moment-closure approximation. We start by writing the number of possible configurations S for a network with a given proportion of knights in each of the states, which yields

$$S = \binom{TL}{[TL]} \binom{\frac{T(T-1)}{2}}{[TT]} \binom{\frac{L(L-1)}{2}}{[LL]},$$

where $\binom{a}{b} = a! / [b!(a-b)!]$ is the binomial coefficient. In this equation the three factors arise from the number of possibilities for placing the TT, TL, and LL links,

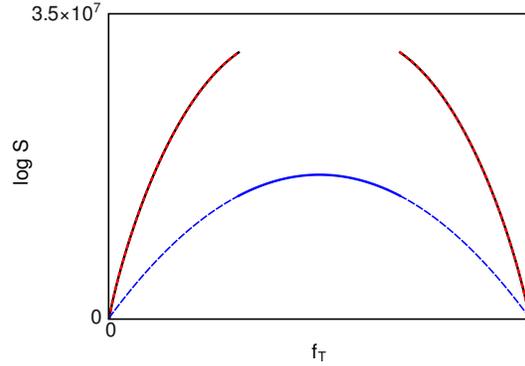


Figure 5. State selection. Shown is the logarithm of the number of microscopic network configurations realizing the different macroscopic states vs. the fraction of T nodes f_T , for a network with $N = 10^4$ nodes, estimated via moment closure approximation. The logarithm of the number of configurations is shown for branches of non-bipartite steady state (black), unstable steady state (red, almost coinciding with the black) and the absorbing bipartite state (blue). Solid lines denote states that are approached by the system.

respectively.

Substituting the steady states from the kinetic model 10 yields the results shown in figure 5. Outside the bipartite regime even the logarithm of the number of states is orders of magnitude larger in the non-bipartite branches than in the bipartite branch. This suggests that the mechanism that drives the system to the non-bipartite state, outside the bipartite regime, can be understood in terms of a *configurational entropy* which is maximized in the observed steady state. This state is also the one that minimizes the ratio between the mean degree of the minority agents to that of the majority agents.

7. Conclusions

In the present paper, we proposed a toy model for nonlocal topological dynamics in a simple network formation game. We showed that this model self-organizes to a completely bipartite state in a wide parameter range, whereas links breaking the bipartiteness appear in other parameter ranges. We explored the genesis of the bipartite regime by network-level simulations, simple thermodynamics arguments, and a detailed kinetic model.

The proposed system showed many characteristics that are closely reminiscent of first-order phase transitions. It may therefore provide an analytically tractable example of a discontinuous phase transition far from equilibrium.

Let us remark that, at present, we cannot conclusively prove that the proposed system meets all criteria that are commonly applied to identify a phase transition. One concern is perhaps that in the kinetic model, the phase transition does not show up as a single discontinuous transition. Instead, the observed order parameter profile emerges due to the presence of *two* discontinuous transitions. However, we note that bifurcations, unlike phase transitions, are not part of physical reality but features of a specific model. The same physical transitions may therefore be described by different

bifurcations in different modeling frameworks.

We are confident that future works will confirm the nature of the transition proposed here. A promising starting point for this work will be refinements of the kinetic theory. The kinetic theory presented here is relatively simple and, in particular, neglects certain long-range correlations. While the theory captures the behavior of the system relatively well, it is likely that deeper insights can be gained by applying more sophisticated approximation schemes. In particular, it is conceivable that this will change the bifurcation diagram turning the discontinuous saddle-node bifurcations into continuous transcritical bifurcations and revealing the spinodal branches of the system.

While more sophisticated approximation schemes, such as higher-order homogeneous approximations or heterogeneous pair-approximations will require significantly more work, we believe that the prospect of having an analytically tractable example of non-equilibrium first-order transitions is well worth this effort.

Acknowledgments

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