Design of Self-Organising Networks

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Abstract—A key problem in the study and design of complex systems is the apparent disconnection between the microscopic and the macroscopic. It is not straightforward to identify the local interactions that give rise to an observed global phenomenon, nor is it simple to design a system that will exhibit some desired global property using only local knowledge. Here we propose a methodology that allows for the identification of local interactions that give rise to a desired global phenomenon of a network, specifically the degree distribution. Given a set of observable processes acting on a network, we determine the conditions that they must satisfy in order to generate a desired steady-state degree distribution. We thereby provide a simple example for a class of tasks where a system can be designed to self-organize to a given state.

Index Terms—Complex networks, network dynamics, self-organization

1 Introduction

Complex systems can exhibit phenomena and properties that are not inherent in the system's constituents but arise from their interactions. In particular, ordered structures can be formed without requiring pre-appointed hubs or leaders [1].

In biology the ability of complex systems to form macroscopic structures and patterns based on simple local rules is evident in all organisms and on all levels of organization. Examples range from the formation of complex (bio)molecules from simple chemical reactions, via the development of tissues and organisms, to social organization and collective decision-making [2].

Many examples of self-organization complex systems can be found in technical systems too, including particular types of power-cuts [3], traffic jams [4], and structural instabilities in constructions [5]. While self-organization is thus essential for the functioning of biological systems, it often appears in technical systems primarily as a source of failure.

The ability of biological systems to exploit self-organization stems from the way in which they have evolved. The process of trial-and-error in biological evolution can discover beneficial local rules. While some degree of trial-and-error is also involved in the development of technical systems, this process is cut short by rational design.

It is tempting to exploit self-organization in technical systems as the biological examples show that self-organizing systems are typically highly resilient. However, our ability to rationally design self-organizing systems is limited by our ability to foresee the macroscopic behaviour to which a given set of local interactions leads. Therefore, self-organization is presently not widely exploited in the functioning of technical systems, and if self-organization takes place in these systems the effect is often disruptive. By advancing our ability to foresee the macroscopic results of local interactions, research in complexity may thus enhance our ability to engineer highly robust technical systems.

A major tool in complex systems research is network modelling [6]–[8]. Depicting a complex system as a network, a set of discrete nodes connected by discrete links, simplifies the constituents of the system but retains the complexity that is inherent to their pattern of interactions. Such models are therefore geared towards analysing the emergence of macroscopic structure and patterns from these interactions.

A macroscopic property that has received particular attention is the degree distribution, the probability distribution of the number of links attached to a randomly drawn node. A challenge is thus to determine to what degree distribution a certain set of local rules leads, or conversely, to create a set of local rules that results in a given degree distribution. Early works addressed this challenge for particular distributions. For instance seminal papers [9], [10] and a more detailed subsequent analysis [11] showed that linear preferential attachment (see below) leads to power-law degree distributions. More recently, progress has been made by a class of methods called heterogeneous moment closure approximations [12]–[16], which cap-

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ture the time evolution of the numbers of certain classes of motif in the system by an infinite system of ordinary differential equations (ODEs). Further, we have shown [17] that the infinite-dimensional ODE systems from heterogeneous approximations can be transformed into a low-dimensional system of PDEs.

In this paper we show how our previously proposed method [17] can be used to design sets of local rules that result in a dynamical network that self-organizes to a given target degree distribution. The proposed method is widely applicable and can be extended to cover other network metrics beyond the degree distribution.

2 METHOD

We address the following challenge: given a set of permissible dynamical processes and a target degree distribution, we seek to determine the rates of processes that drive the system to the target degree distribution. The proposed method can be broken into steps as follows:

1) Describe the evolution of the network using a heterogeneous approximation. This leads to an infinite system of ODEs that describe the temporal evolution of the elements of the degree distribution p_k.
2) Transform the infinite system of ODEs obtained from the heterogeneous approximation into a first-order PDE for the generating function $G(x) = \sum_k p_k x^k$.
3) Transform the desired steady-state degree distribution into its generating function form and substitute into the PDE.
4) Use the resulting expression to determine whether the degree distribution is possible and, if so, obtain the relation between rates that must hold.

3 SELF-ORGANISATION WITH FIXED PROCESS RATES

We begin by focusing on the self-organization of networks through processes for which the rate per node or per link (depending on the process) is constant. Considering a finite set of such processes constrains the degree distributions that can be evolved. In this setting the proposed method provides a test that determines whether a desired degree distribution can be created by a given set of processes or not. If the distribution can be created then the method reveals the relative rates of processes. We illustrate this procedure in four examples: the Poissonian degree distribution, which we mainly use as an illustrative example, the scale-free, negative binomial and geometric distributions.

We begin by considering a network of discrete nodes connected by unweighted, undirected links (labelled $i - j$, for a link between nodes i and j). We assume that the network changes due to the following eight processes which occur stochastically in time.

- *Random rewiring*. A link $i - j$ is selected at random, i.e. with uniform distribution, and broken. One of the two formerly connected nodes $a \in \{i, j\}$ is chosen randomly with equal probability, and a new link created between a and a target node b, where b is chosen randomly from all the nodes in the network that are not currently a neighbour of a. The rate (per link) at which random rewiring occurs is w_{r}.
- *Preferential rewiring*. A randomly selected link $i - j$ is broken and one of the two formerly connected nodes $a \in \{i, j\}$ is chosen randomly with equal probability. A new link is created between the chosen node and a target node b, not currently connected to a. For the target node b we preferentially select nodes of high degree, such that the probability of a node being chosen increases linearly with the degree of the node. The rate (per link) at which preferential rewiring occurs is w_{p}.
- *Deletion of links*. A randomly selected link $i - j$ is chosen from the network and deleted. The rate (per link) for the removal of links is l_{d}.
- *Random addition of links*. Two unconnected nodes i and j are picked randomly from the network and a link $i - j$ is formed between them. The rate (per node) at which random addition of links occurs is l_{r}.
- *Preferential addition of links*. Two unconnected nodes i and j are chosen from the network and a link $i - j$ is formed between them. Both nodes are chosen preferentially, with the probability proportional to the degree of the node. The rate (per node) at which preferential addition of links occurs is l_{p}.
- *Deletion of nodes*. A node is selected at random from the network and deleted, together with all its links. The rate (per node) for the removal of nodes is n_{d}.
- *Random addition of nodes*. A node of (fixed) degree m is added to the network. The incoming node forms links to m existing nodes in the network, which are chosen at random. The rate (per node) at which random addition of nodes occurs is n_{r}.
- *Addition of nodes by preferential attachment*. A node of (fixed) degree m is added to the net-

TABLE 1
Target Degree Distributions Produced Using Fixed Process Rates

Target distribution	$p_k^\star$	$G^\star(x)$	Rates
Poisson	$\frac{\mathrm{e}^{-\langle k\rangle}\langle k\rangle^k}{k!}$	$\mathrm{e}^{\langle k\rangle(x-1)}$	$\langle k\rangle = \frac{2l_\mathrm{r}}{l_\mathrm{d}} \quad w_\mathrm{r}=c$ $w_\mathrm{p}=l_\mathrm{p}=n_\mathrm{r}=n_\mathrm{p}=n_\mathrm{d}=0$
Power-law	0 if $k<m$ $\frac{2m(m+1)}{k(k+1)(k+2)}$ if $k\geq m$	$\sum_{k\geq m}\frac{2m(m+1)}{k(k+1)(k+2)}x^k$	$n_\mathrm{p}=c$ $l_\mathrm{d}=l_\mathrm{p}=l_\mathrm{r}=0$ $w_\mathrm{p}=w_\mathrm{r}=n_\mathrm{r}=n_\mathrm{d}=0$
Negative-binomial	$\binom{k+r-1}{k}p^k(1-p)^r$	$\left(\frac{1-p}{1-px}\right)^r$	$p=\frac{\langle k\rangle w_\mathrm{p}+2l_\mathrm{p}}{\langle k\rangle(w_\mathrm{r}+w_\mathrm{p}+l_\mathrm{d})}$ $r=\frac{\langle k\rangle(\langle k\rangle w_\mathrm{r}+2l_\mathrm{r})}{\langle k\rangle w_\mathrm{p}+2l_\mathrm{p}}$ $n_\mathrm{r}=n_\mathrm{p}=n_\mathrm{d}=0$
Geometric	$p(1-p)^k$	$\frac{p}{1-(1-p)x}$	$p=\frac{\langle k\rangle(w_\mathrm{r}+l_\mathrm{d})-2l_\mathrm{p}}{\langle k\rangle(w_\mathrm{r}+w_\mathrm{p}+l_\mathrm{d})}$ (subject to the condition) $0=\langle k\rangle^2 w_\mathrm{r}+\langle k\rangle(2l_\mathrm{r}-w_\mathrm{p})-2l_\mathrm{p}$ $n_\mathrm{r}=n_\mathrm{p}=n_\mathrm{d}=0$

work. The incoming node forms links to m existing nodes in the network which are chosen preferentially, with probability proportional to their degree. Hence nodes of higher degree are more likely to form links with the incoming node than nodes of lower degree. The rate (per node) at which preferential addition of nodes occurs is n_p.

Our goal is to determine rates for the different processes, such that the network degree distribution p_k approaches a target p_k^*. Using a heterogeneous mean field approximation [18] we derive the evolution equation for the degree distribution p_k $(k \in \mathbb{N})$

$$
\begin{aligned}
\frac{\mathrm{d}p_k}{\mathrm{d}t} &= w_\mathrm{r}\left[(k+1)p_{k+1} - kp_k\right. \\
&\qquad \left. + \left(\textstyle\sum_{k'} k'p_{k'}\right)(p_{k-1}-p_k)\right] && \text{(1i)}\\
&+ w_\mathrm{p}\left[((k+1)p_{k+1}-kp_k)\right. \\
&\qquad \left. + ((k-1)p_{k-1}-kp_k)\right] && \text{(1ii)}\\
&+ l_\mathrm{d}\left[(k+1)p_{k+1}-kp_k\right] && \text{(1iii)}\\
&+ 2l_\mathrm{r}\left[p_{k-1}-p_k\right] && \text{(1iv)}\\
&+ 2l_\mathrm{p}\left[\left(1/\textstyle\sum_{k'} k'p_{k'}\right)((k-1)p_{k-1}-kp_k)\right] && \text{(1v)}\\
&+ n_\mathrm{d}\left[(k+1)p_{k+1}-kp_k\right] && \text{(1vi)}\\
&+ n_\mathrm{r}\left[m(p_{k-1}-p_k)-p_k+\delta_{m,k}p_k\right] && \text{(1vii)}\\
&+ n_\mathrm{p}\left[\left(m/\textstyle\sum_{k'} k'p_{k'}\right)((k-1)p_{k-1}-kp_k)\right. \\
&\qquad \left. -p_k+\delta_{m,k}p_k\right], && \text{(1viii)}
\end{aligned}
$$

where $\{w_\mathrm{r}, w_\mathrm{p}, l_\mathrm{d}, l_\mathrm{r}, l_\mathrm{p}, n_\mathrm{d}, n_\mathrm{r}, n_\mathrm{p}\}$ are the rates that we seek to determine, and $\delta_{m,k}$ is the Kronecker delta.

Each line of (1) corresponds to one of the processes. The different terms correspond to different effects of one process. For instance, (1i) describes random rewiring. The term proportional to $(k+1)p_{k+1} - kp_k$ captures the effect of links being rewired away; the first term represents the gain in nodes of degree k because of nodes of degree $k+1$ losing one link, while the second represents the loss of nodes of degree k due to such nodes losing one link. The term proportional to $\left(\sum_{k'} k'p_{k'}\right)(p_{k-1}-p_k)$ captures the effect of links being rewired to a node, where the summation is necessary because the rate depends on the total number of links that are rewired.

The heterogeneous expansion thus results in an infinite system of ODEs, which we transform into a first-order quasilinear PDE by use of generating functions [19]. We start by defining the generating function $G(x,t) = \sum_k p_k(t)x^k$. The underlying idea of this transformation is to interpret the elements of the degree distribution as coefficients of a Taylor series of a function G in an arbitrary variable x. This transformation is advantageous because it allows us to work with the continuous object G rather than the discrete set p_k. Because the transformation is reversible (by a Taylor expansion of G) no information is lost in the transformation. Thus investigating the time evolution of G reveals the same information as investigating the time evolution of p_k.

To study the time dependence of G we multiply (1) by x^k and sum over $k \geq 0$ yielding a first-order PDE for $G(x,t)$

$$\begin{aligned} G_t = (x-1)\left[x\left(w_\mathrm{p}+\frac{2l_\mathrm{p}}{G_x(1,t)}+\frac{n_\mathrm{p}m}{G_x(1,t)}\right)\right. \\ \left. -w_\mathrm{r}-w_\mathrm{p}-l_\mathrm{d}-n_\mathrm{d}\right]G_x \\ +\left[(x-1)\left(w_\mathrm{r}G_x(1,t)+2l_\mathrm{r}+n_\mathrm{r}m\right)\right. \\ \left. -\ n_\mathrm{r}-n_\mathrm{p}\right]G+\left(n_\mathrm{r}+n_\mathrm{p}\right)x^m \end{aligned} \quad (2)$$

where $G_t = \partial G/\partial t$, etc.

To arrive at this equation we broke the right hand side summation into individual sums and then shifted the summation index to turn all instances of p_{k+1} and p_{k-1} into p_k. Factors of x can be pulled into or out of the sums as necessary, while factors of k are eliminated using the fact that $\sum kp_kx^{k-1} = \partial_x \sum p_kx^k = G_x$ [19] leading to the appearance of the spatial derivative in (2). Finally we used $G_x(1,t) = \sum_k kp_k(t)$ to eliminate the sums that appear in (1).

We note that the PDE (2) describes the dynamics in a space spanned by x, the abstract variable that we introduced to write the generating function, which is hence devoid of physical meaning. Because of this lack of physical interpretation, there are (fundamentally) no boundary conditions for this PDE.

In the present paper we do not attempt to solve this PDE but only seek to determine under which conditions it admits a desired solution. Given a target degree distribution $p^\star_k$ we can compute the corresponding target generating function

$$G^\star(x) = \sum_k p^\star_k x^k.$$

Substituting $G = G^\star(x)$ into (2) we obtain an algebraic condition that must be met in order for the system to permit the desired degree distribution as a stationary solution.

For a simple demonstration we first consider the Poisson distribution $p^\star_k = \exp(-\langle k\rangle)\langle k\rangle^k/k!$ [20] as our target distribution, where $\langle k\rangle$ is the target distribution mean degree. Since the Poisson degree distribution is the degree distribution of a completely random graph, one can guess that this distribution can be created by random rewiring of links or by random addition and deletion of links. To show this using the proposed method we compute the target generating function

$$G^\star(x) = \mathrm{e}^{-\langle k\rangle}\sum_k \frac{\langle k\rangle^k x^k}{k!} = \mathrm{e}^{\langle k\rangle(x-1)}. \quad (3)$$

Substituting (3) into (2) yields

$$\begin{aligned} 0 = (x-1)\left[x\left(w_\mathrm{p}+\frac{2l_\mathrm{p}}{\langle k\rangle}+\frac{n_\mathrm{p}m}{\langle k\rangle}\right)\right. \\ \left. -w_\mathrm{r}-w_\mathrm{p}-l_\mathrm{d}-n_\mathrm{d}\right]\langle k\rangle\,\mathrm{e}^{\langle k\rangle(x-1)} \\ +\left[(x-1)\left(w_\mathrm{r}\langle k\rangle+2l_\mathrm{r}+n_\mathrm{r}m\right)\right. \\ \left. -n_\mathrm{r}-n_\mathrm{p}\right]\mathrm{e}^{\langle k\rangle(x-1)}+\left(n_\mathrm{r}+n_\mathrm{p}\right)x^m. \end{aligned} \quad (4)$$

which must hold for all $x \in \mathbb{R}$. Thus the coefficients of the linearly independent functions $x^2\exp(-\langle k\rangle(x-1))$, $x\exp(-\langle k\rangle(x-1))$, $\exp(-\langle k\rangle(x-1))$, and x^m must all be zero. In particular, then, since the coefficient of x^m must be zero and the rates must be non-negative, we have that $n_\mathrm{r} = n_\mathrm{p} = 0$. This implies that there can be no addition of nodes to the network, and hence the rate for removal of nodes must also be zero ($n_\mathrm{d} = 0$) to prevent an absorbing state of an empty network. Under these conditions, (4) simplifies to

$$\begin{aligned} 0 = \langle k\rangle\left[x\left(w_\mathrm{p}+\frac{2l_\mathrm{p}}{\langle k\rangle}\right)-w_\mathrm{r}-w_\mathrm{p}-l_\mathrm{d}\right] \\ +\left[(w_\mathrm{r}\langle k\rangle+2l_\mathrm{r})\right]. \end{aligned} \quad (5)$$

which gives two equations for the remaining rates (as above, the coefficients of the linearly independent functions of x must be zero). These yield $w_\mathrm{p} = l_\mathrm{p} = 0$, $\langle k\rangle = 2l_\mathrm{r}/l_\mathrm{d}$, and $w_\mathrm{r} = c$, any constant. Since the number of links and nodes remains constant for rewiring, the random rewiring rate does not affect the mean degree of the network. Hence $G^\star(x) = \exp\left[2l_\mathrm{r}/l_\mathrm{d}(x-1)\right]$, and the mean degree is the ratio of the rates governing random link addition and link removal.

As expected, the results show that it is possible to design a network with a steady state Poisson distribution with any desired mean degree by choosing rates for random link addition and random link deletion, with a specific quotient. If $l_\mathrm{r} = l_\mathrm{d} = 0$ and the only process acting on the network is random rewiring, then the mean degree remains the same as the initial mean degree of the network, and so $G^\star(x) = \exp\left[\langle k\rangle(x-1)\right]$ where $\langle k\rangle$ is the initial mean degree.

Sets of rates that let the network self-organize to other degree distributions can be identified analogously. We cannot expect to be able to create an arbitrary degree distribution from a finite set of processes running at constant rates. However, already the set of eight processes considered so far allows us to design networks that self-organise to several common statistical distributions. We present an overview of some examples in Table 1 and discuss them briefly below.

It is well known that power-law degree distributions with exponent $\gamma = 3$ emerge from a process of

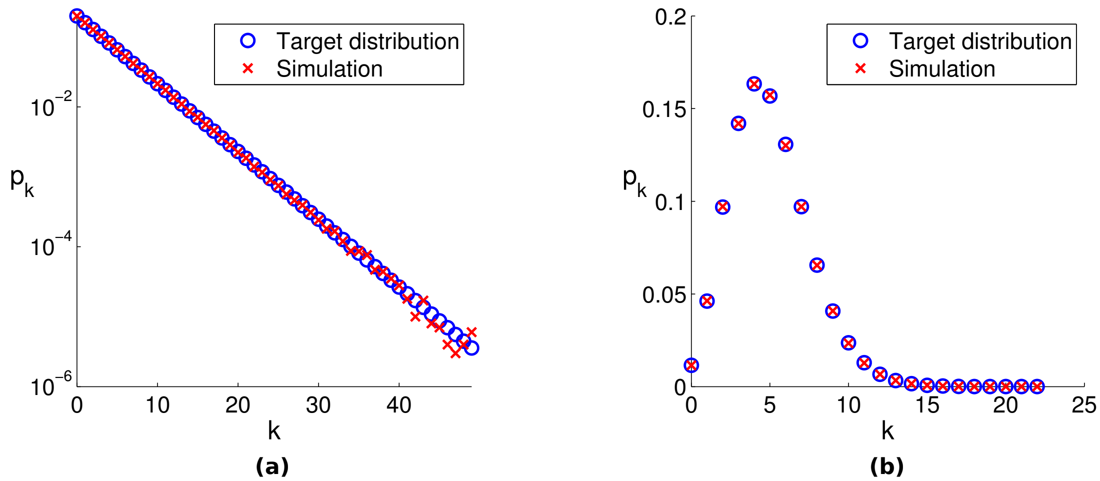

Fig. 1. Self-organising networks with local rules achieve target degree distributions. Shown are the target degree distributions (circles) and the self-organised degree distributions in agent-based simulations (crosses). (a) The target distribution is long-tailed with $r = 1$,$p = 0.8$ and $\langle k\rangle = 4$. Hence preferential processes dominate and the corresponding process rates for the simulation are $l_{\mathrm{r}} = 0.01$, $l_{\mathrm{p}} = 0.04$ and $l_{\mathrm{d}} = 0.025$ with all other rates zero (b) The target distribution is more Poissonian with $r = 20$, $p = 0.2$ and $\langle k\rangle = 5$. Hence random processes dominate and the corresponding process rates for the simulation are $l_{\mathrm{r}} = 0.04$, $l_{\mathrm{p}} = 0.01$ and $l_{\mathrm{d}} = 0.02$ with all other rates zero.

preferential attachment [9]. Repeating the procedure above with the same set of processes, for such a desired power-law degree distribution, reveals that a network subject to these processes running at constant rates, can only approach a power law degree distribution when addition of nodes by preferential attachment is the only process with non-zero rate.

The negative-binomial degree distribution [21] has two free parameters p and r. When $r = 1$ we have a geometric distribution, and when $r \to \infty$ we recover a Poisson distribution. Applying the proposed method reveals the dependence of the parameters p and r on the rates of processes, shown in Table 1, and shows that the distribution is possible whenever there is no addition or deletion of nodes ($n_{\mathrm{d}} = n_{\mathrm{r}} = n_{\mathrm{p}} = 0$). In this case we have five free parameters to meet the two conditions that arise from the method, in order to obtain a network with desired values of p and r. Furthermore, $\langle k\rangle = G_x(1)$, and so we can determine $\langle k\rangle$ in terms of p and r, and hence the process rates. Substituting this relationship into the results for p and r from Table 1 yields

$$\begin{aligned} p &= \frac{(l_d + w_p)\, l_p + l_r w_p}{(l_r + l_p)\,(w_r + w_p + l_d)}, \\ r &= \frac{2(l_r + l_p)(l_p w_r + l_r(w_r + l_d))}{l_d(l_r w_p + l_p(l_d + w_p))}, \\ \langle k\rangle &= \frac{2(l_r + l_p)}{l_d}. \end{aligned} \tag{6}$$

Alternatively, in the case $l_d = l_r = l_p = 0$, where links are neither deleted nor added, $\langle k\rangle$ is equal to the initial mean degree of the network, and hence an additional free parameter, resulting in

$$p = w_p/(w_p + w_r), \qquad r = \langle k\rangle w_r/w_p. \tag{7}$$

It is therefore possible to produce a specific steady state with a desired p, r (and possibly $\langle k\rangle$) by choosing rates to satisfy either (6) or (7). For purposes of illustration, we choose parameter values that typify the different classes of distribution exhibited by the negative binomial. For parameter sets such as $r = 1$, $p = 0.8$ (and hence $\langle k\rangle = 4$), the target negative-binomial distribution is long-tailed, which needs strong preferential addition of links. Based on (6) we choose the rates to be $l_{\mathrm{r}} = 0.01$, $l_{\mathrm{p}} = 0.04$ and $l_{\mathrm{d}} = 0.025$, with all other rates zero. On the other hand, for distributions where r is large, such as $r = 20$, $p = 0.2$ (and hence $\langle k\rangle = 5$), the distribution is more Poisson-like, and we need strong random addition of links. To achieve this we chose rates $l_{\mathrm{r}} = 0.04$, $l_{\mathrm{p}} = 0.01$ and $l_{\mathrm{d}} = 0.02$, with all other rates zero, again using (6). Fig. 1 compares the results of agent based simulations subject to these rates with the desired target distributions. In Fig. 1(a) we have the long-tailed distribution, while Fig. 1(b) shows the Poisson-like distribution. The simulation results are in good agreement with the target distribution; the discrepancy at high degree in Fig. 1(a) is due to the infrequency of nodes with high degree. This would approach the desired target as the size of the simulation increases.

The final example in Table 1 is the geometric distribution [21], which has one free parameter p. Using the proposed method gives an expression for p in terms of the process rates and one additional condition that must be satisfied in order to produce a geometric distribution. Again we have more processes than constraints so many different combinations are possible. If we have just rewiring processes acting on the network, the mean degree of the network will equal the initial mean degree, and we find two relationships between the rewiring rates: $p = w_{\mathrm{r}}/(w_{\mathrm{r}} + w_{\mathrm{p}})$, and $\langle k\rangle w_{\mathrm{r}} = w_{\mathrm{p}}$. Alternatively, with only addition and deletion of links, we can again derive the mean degree $\langle k\rangle$ from the generating function, in terms of the process rates, using the fact that $\langle k\rangle = G_x(1)$. We thus find $\langle k\rangle = (1-p)/p = (l_{\mathrm{r}} + l_{\mathrm{p}})/l_{\mathrm{d}}$ and hence the two conditions for the rates are $p = (l_{\mathrm{d}} - l_{\mathrm{r}})/l_{\mathrm{d}}$ and $l_{\mathrm{r}}(l_{\mathrm{r}} + l_{\mathrm{p}}) = l_{\mathrm{p}} l_{\mathrm{d}}$.

We have shown that it is possible to produce a number of different degree distributions using the processes of random and preferential rewiring, random and preferential link and node addition, and link and node removal. Clearly there are also many distributions that cannot be obtained with the rules considered so far, where applying the proposed method yields conditions that do not admit any solution. In such a case we have two options. First, we can expand the set of processes by allowing one or more additional processes. Analysing the (unsolvable) conditions obtained from the initial rule set should give us a good idea which additional processes are necessary to admit a solution. Second, we can relax the assumption that processes run at constant rates, which we discuss in the next section.

4 Networks with degree-dependent rates

Up to this point we have assumed that processes occur at constant rates (per node or per link) that are independent of the respective node's or link's properties. By contrast, in many systems studied in nature rates depend on node properties, such as the node's degree. Also, in technical applications it is easily conceivable that the nodes are aware of their own degree and take it into account in their behaviour. We therefore consider degree-dependent rates in the context of the method proposed here.

Allowing degree-dependent rates greatly increases the range of degree distributions that can be obtained with a given number of processes, which enables us to restrict the set of processes considered. For illustration we only consider degree-dependent link creation and deletion.

Two variants of degree-dependent link creation/deletion processes are conceivable: using either non-local or local information. In the first, non-local, variant the task of creating a network with given degree distribution is trivial, as we end up with the configuration model [22]. Furthermore, the non-local variant requires non-local knowledge to be available at each node and is hence infeasible in many technical applications. We therefore do not consider the non-local degree-dependent processes here.

Instead we consider local degree-dependent link creation and deletion processes. In this local variant, the decision to create or delete a link is made by the nodes independently, taking only their own degree into account. If a node decides to delete a link it chooses the link randomly among its existing links. If a node decides to create a link it establishes the link to another node that is randomly selected from the whole population. Thus nodes are also subject to link creation and deletion events by partners, which are not under their control.

Since we are only considering link creation and deletion events the time evolution of the degree distribution p_k is captured by

$$\frac{\mathrm{d}p_k}{\mathrm{d}t} = -l_k p_k + l_{k+1} p_{k+1} \tag{8i}$$

$$+ \frac{\sum_k p_k l_k}{\sum_k k p_k}\left[(k+1)p_{k+1} - k p_k\right] \tag{8ii}$$

$$- m_k p_k + m_{k-1} p_{k-1} \tag{8iii}$$

$$+ \sum_k p_k m_k \left[p_{k-1} - p_k\right]. \tag{8iv}$$

The terms in (8) describe the change in p_k due to the removal of links at a rate l_k and addition of links at a rate m_k. Terms (8i) and (8iii) are due to the focal node, of degree k, having a link deleted/added, while (8ii) and (8iv) are due to a neighbour, of any degree, adding or removing a link to the focal node.

We now define three generating functions. The first is the generating function for the degree distribution p_k, $G(x,t) = \sum_k p_k(t) x^k$, while the remaining two represent the degree distribution multiplied by the link removal rate, $S(x,t) = \sum_k l_k p_k(t) x^k$ and the link addition rate, $T(x,t) = \sum_k m_k p_k(t) x^k$. The need to define two new generating functions stems from the non-constant process rates; when these rates are multiplied by the degree distribution the result will not in general be a multiple of the generating function G. The form of the new generating functions is chosen to make the transformation of (8) to a generating function PDE straightforward. Multiplying (8) by x^k and summing over $k \geq 0$ gives the first-order PDE

$$G_t = S\left(\frac{1}{x} - 1\right) + T(x-1)$$

TABLE 2
Target Degree Distributions Produced Using Degree Dependent Rates

Target distribution	$p_k^\star$	$G^\star(x)$	Rates
Poisson	$\frac{e^{-\langle k \rangle}\langle k \rangle^k}{k!}$	$e^{\langle k \rangle (x-1)}$	$l_k = \frac{m_{k-1}k}{\langle k \rangle}$
	$\frac{(k+1)e^{-a}a^k}{(1+a)k!}$	$\frac{1+ax}{1+a}e^{a(x-1)}$	$l_k = \frac{k^2(m_{k-1}+\bar{T})}{a(k+1)} - \frac{k\bar{T}}{\langle k \rangle}$
Power-law	c if $k=0$; $(1-c)k^{-\alpha}/\zeta(\alpha)$ if $k \geq 1$	$c + \frac{(1-c)\mathrm{Li}_\alpha(x)}{\zeta(\alpha)}$	$l_0 = 0$; $l_1 = \frac{c\zeta(\alpha)(m_0+\bar{T})}{1-c} - \frac{\bar{T}}{\langle k \rangle}$; $l_k = \frac{(k-1)^{-\alpha}(m_{k-1}+\bar{T})}{k^{-\alpha}} - \frac{k\bar{T}}{\langle k \rangle}$, for $k \geq 2$
Bimodal	$\frac{e^{-a}a^k + e^{-b}b^k}{2k!}$	$\frac{1}{2}\left(e^{a(x-1)} + e^{b(x-1)}\right)$	$l_k = \frac{k(m_{k-1}+\bar{T})\left(e^{-a}a^{k-1} + e^{-b}b^{k-1}\right)}{\left(e^{-a}a^k + e^{-b}b^k\right)} - \frac{k\bar{T}}{\langle k \rangle}$

$$+\frac{S(1)}{G_x(1)}(1-x)\,G_x + T(1)\,(x-1)\,G. \quad (9)$$

In the steady state this simplifies to

$$S = x\left(T + \bar{T}G - \frac{\bar{S}G_x}{\langle k \rangle}\right), \quad (10)$$

where $\bar{S} = S(1)$ is the total rate of link addition events per node, $\bar{T} = T(1)$ is the total rate of link deletion events, and $\langle k \rangle$ is the mean degree as above.

Since we do not consider node additions or deletions, the degree distribution can only be stationary if the total link addition and deletion rates are identical. We can verify this by evaluating (9) at $x = 1$. Since $G_x(1) = \langle k \rangle$, $T(1) = \bar{T}$ and $S(1) = \bar{S}$ we find $\bar{T} = \bar{S}$ as expected.

As before, we have a great deal of freedom when specifying the link rates. Typically one first chooses m_k which in turn determines l_k, where one must be careful to check that the particular choice of m_k does not result in negative values for l_k.

For simplicity, we again consider which combinations of processes can lead to the Poisson distribution, which has desired degree distribution $p_k^\star = \exp(-\langle k \rangle)\langle k \rangle^k/k!$, and hence $G^\star(x) = \exp\left[\langle k \rangle(x-1)\right]$. Substituting $G = G^\star(x)$ into (10) yields

$$S = x\left[T + (\bar{T} - \bar{S})\,G^\star\right]. \quad (11)$$

Since $\bar{S} = \bar{T}$, we can cancel the two terms in (11) and are left with the relationship $S = xT$. By comparing coefficients of x^k we find the condition $l_k = m_{k-1}(p_{k-1}/p_k)$ and hence $l_k = km_{k-1}/\langle k \rangle$.

We can use this relationship to reproduce a result from the previous section. If links are added independently of degree, e.g. $m_k = 1$, the required loss rate is $l_k = k/\langle k \rangle$. So links are lost proportionally to a node's degree, which means a fixed-rate link loss per link, which leads to the same system identified above.

This solution is not unique. For example, if we allow links to be added at a rate proportional to degree, so $m_k = k$, then $l_k = k(k-1)/\langle k \rangle$, such that loss is proportional to the number of distinct pairs of links connecting to a node.

The above analysis can be repeated with other distributions. Some examples are listed in Table 2 (where $\zeta(\alpha)$ is the zeta function and $\mathrm{Li}_\alpha(x)$ is the polylogarithm of x). Once we have a relationship between l_k and m_k, as given in Table 2 , we can choose values for m_k (or l_k) and hence calculate $\bar{T}$ in order to find the corresponding l_k (or m_k).

For example, Table 2 gives the condition for a power law degree distribution with exponent α and given $p_0 = c$ to prevent divergence of the distribution at $k = 0$. A comparison between the target distribution, with $\alpha = 2.5$ and $c = 0.5$, and an agent-based simulation is shown in Fig. 2(a). Using the rules in Table 2 we simulate the network by adding links to nodes at a rate proportional to their degree, choosing $m_k = 0.02k$, thus in accordance with the conditions in Table 2 delete links at the rates,

$$l_1 = \frac{0.02c\langle k \rangle\zeta(\alpha)}{1-c} - 0.02,$$
$$l_k = \frac{0.02(k-1)^{-\alpha}(k-1+\langle k \rangle)}{k^{-\alpha}} - 0.02k, \;\; k \geq 2.$$

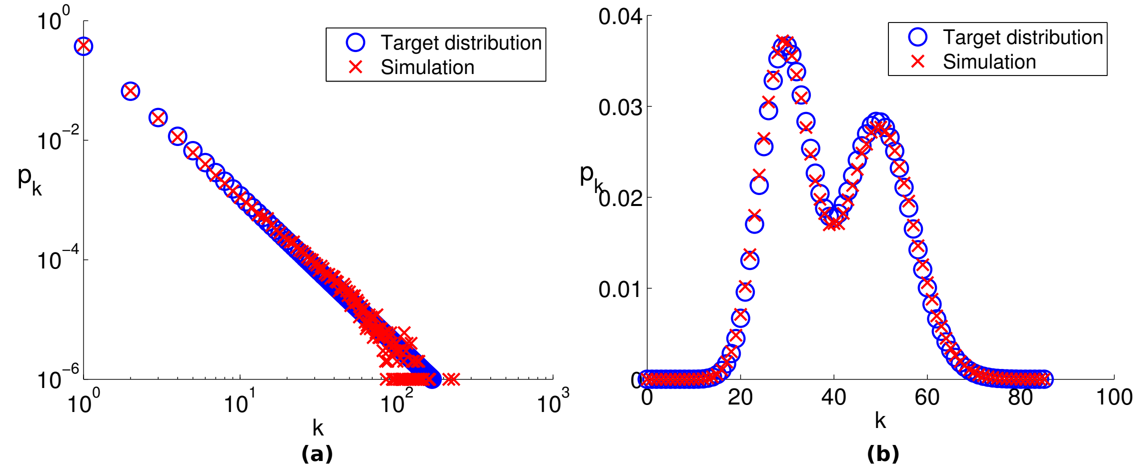

Fig. 2. Self-organizing networks with degree-dependent process rates. Using only link creation and link removal, functional-forms for the degree-dependence of rates were designed such that the network approaches a power-law degree distribution with exponent -2.5 (a), and a bimodal degree distribution with $\langle k \rangle = 40$ (b). Agent-based simulations (crosses) show that the designed system approximate the target distributions (circles).

The results from the agent-based simulation are in good agreement with the desired steady-state degree distribution, where the discrepancy at higher degree is due to the infrequency of nodes with high degree. This discrepancy will decrease as the size of the simulation increases.

The examples so far have all been of unimodal degree distributions, but multimodal distributions can also be achieved. We give an example of a bimodal distribution in Table 2. A comparison between a target bimodal distribution, where $a = 30$ and $b = 50$, and an agent-based simulation using the rules from Table 2 is shown in Fig. 2(b). We add links at a constant rate, $m_k = 0.1$, and hence delete links at a rate

$$l_k = \frac{0.2k\left(\mathrm{e}^{-30}\,30^{k-1} + \mathrm{e}^{-50}\,50^{k-1}\right)}{\mathrm{e}^{-30}\,30^{k} + \mathrm{e}^{-50}\,50^{k}} - 0.0025k.$$

The results are again in good agreement with the desired degree distribution.

5 State-change processes

In applications, the self-organisation of a dynamical network may involve the assignment of functional roles to the nodes. For instance one can imagine a self-organizing sensor network [23], where initially identical smart sensors differentiate into two functional states, say primary recorders of data and aggregators, who integrate data from different recorders and transmit results. In this case we may want the system to evolve a communication network where the aggregators are hubs that connect to many recorders and some other aggregators.

In this section we address the challenge of designing a self-organizing network where both the states of nodes and the state-dependent degree distributions approach predefined targets. We proceed as before and define a set of processes acting on the network and state-dependent degree distributions and frequencies of the different states. We then describe the evolution of the network using a heterogeneous active-neighbourhood approximation [12], [15], [16], which tracks the evolution of nodes in a specific state and the number of neighbours it has in each state. The active-neighbourhood approximation results in coupled infinite-dimensional systems of ODEs, which we then convert into coupled PDEs using generating functions.

For a system with N distinct node states we obtain a system of N coupled PDEs. Even for systems with several states this does not pose a fundamental problem as we do not need to solve the PDEs. By substituting the target degree distributions into the PDE system we find the conditions that the process rates must satisfy to reach the desired target.

For illustration we consider a challenge inspired by the sensor network example. Our aim is to determine rules that self-organize the network to a state where a given proportion of the nodes become aggregators, state A, while the others become recorders, state B. Furthermore we want the aggregators connected among themselves in a network with a Poissonian degree distribution with a desired mean, similarly for the recorder to recorder connections and the aggregator to recorder connections.

We define six dynamical processes acting on the

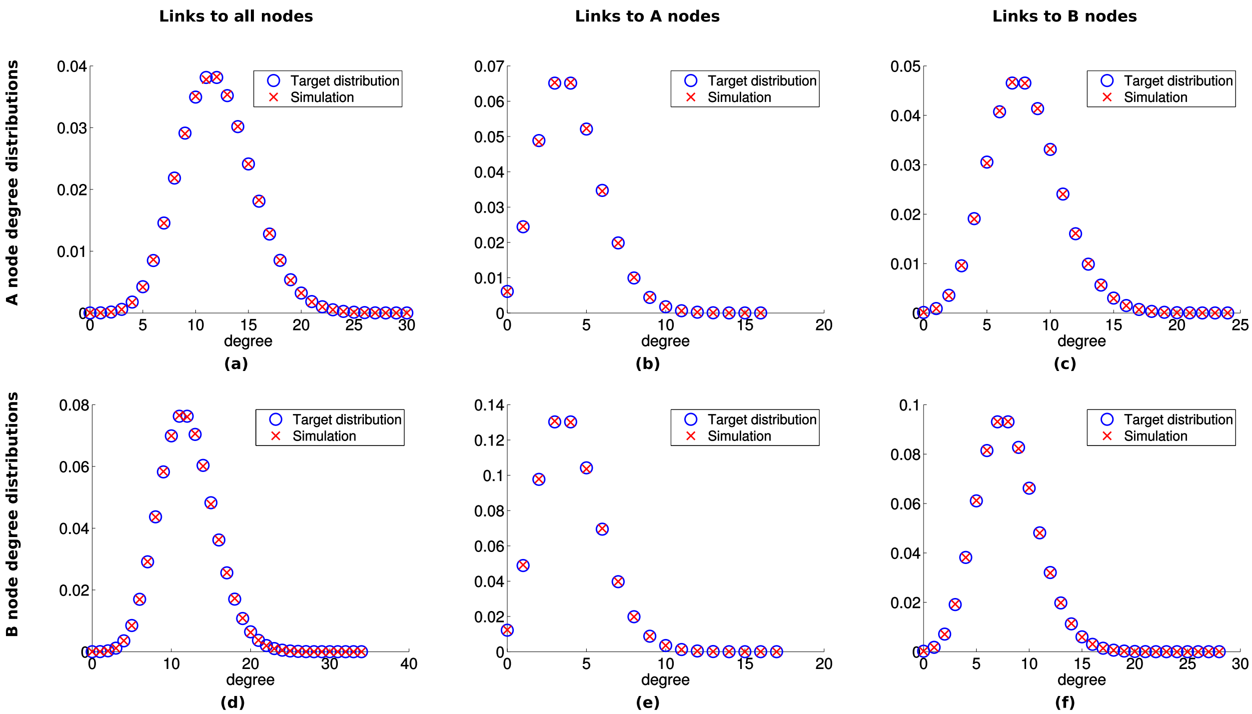

Fig. 3. Local rules generate target distributions in a two-state network. Shown are target distributions (circles) compared to agent-based simulations (crosses) designed to self-organize to the target distribution by using the relations (12). Rates are as follows: $w_{\mathrm{AB-AA}} = 0.01, w_{\mathrm{AB-BB}} = 0.02, w_{\mathrm{AA-AB}} = 0.04, w_{\mathrm{BB-AB}} = 0.02, p_{\mathrm{A-B}} = 0.03, p_{\mathrm{B-A}} = 0.015$. Top is the degree distribution of A-nodes where (a) is the total degree distribution (b) is the degree distribution to A-nodes only (c) is the degree distribution to B-nodes only. Bottom is the degree distribution of B-nodes where (d) is the total degree distribution (e) is the degree distribution to A-nodes only (f) is the degree distribution to B-nodes only.

network comprising link-rewiring and state-change processes, with constant rates w_{p} and p_{p} for a process p respectively, as described below. A node in state $i \in \{A, B\}$ can rewire an existing link from a neighbour in state $j \in \{A, B\}$ to a node in the other state $\bar{j}$, picked uniformly at random from the network. There are four such rewiring processes; the rates at which they occur are denoted as $w_{\mathrm{ij-i\bar{j}}}$. The remaining two processes are state-change processes; a node in state $i \in \{A, B\}$ can switch to the opposite state $\bar{i}$, the rate at which these processes occur are $p_{\mathrm{i-\bar{i}}}$.

We define $N_{k,l}$ as the density of nodes in state $N \in \{A, B\}$ with k A-neighbours, and l B-neighbours. The evolution of the density of $A_{k,l}$ nodes and $B_{k,l}$ nodes under the six processes described above results in two coupled infinite-dimensional systems of ODEs, the equations are given in the online supplementary material in Appendix A.

We next introduce the generating functions $G^A = \sum_{k,l} A_{k,l} x^k y^l$ and $G^B = \sum_{k,l} B_{k,l} x^k y^l$, and convert the pair of infinite-dimensional systems of ODEs into two first-order coupled PDEs; the equations are given in Appendix B. We substitute target steady-state degree distributions into the steady-state generating function equations and compare coefficients of linearly independent functions, as before, in order to find the necessary relationships between rates.

In our sensor network example, we thus define two generating functions: one for the aggregators $G^A(x, y) = c_1 \exp[a(x-1) + b(y-1)]$, and one for the recorders $G^B(x, y) = c_2 \exp[a(x-1) + b(y-1)]$. The exponents a and b are common between G^A and G^B for simplicity; we shall relax this constraint below. Here c_1 is the proportion of aggregators and c_2 is the proportion of recorders, and hence $c_1 + c_2 = 1$ is the total density of sensors. The average number of aggregator to aggregator (or aggregator to recorder) connections per aggregator (or recorder) is a, while the average number of recorder to aggregator (or recorder to recorder) connections per aggregator (or recorder) is b. The choice of values for a, b, c_1 and c_2 is constrained by the condition $c_1 b = c_2 a$, which ensures symmetry; the number of AB-links must be equal to the number of BA-links; this can be equivalently written as $G^A_y(1, 1) = G^B_x(1, 1)$.

Following the proposed method, we substitute G^A and G^B into the steady-state generating function equations in Appendix B. We are able to cancel the

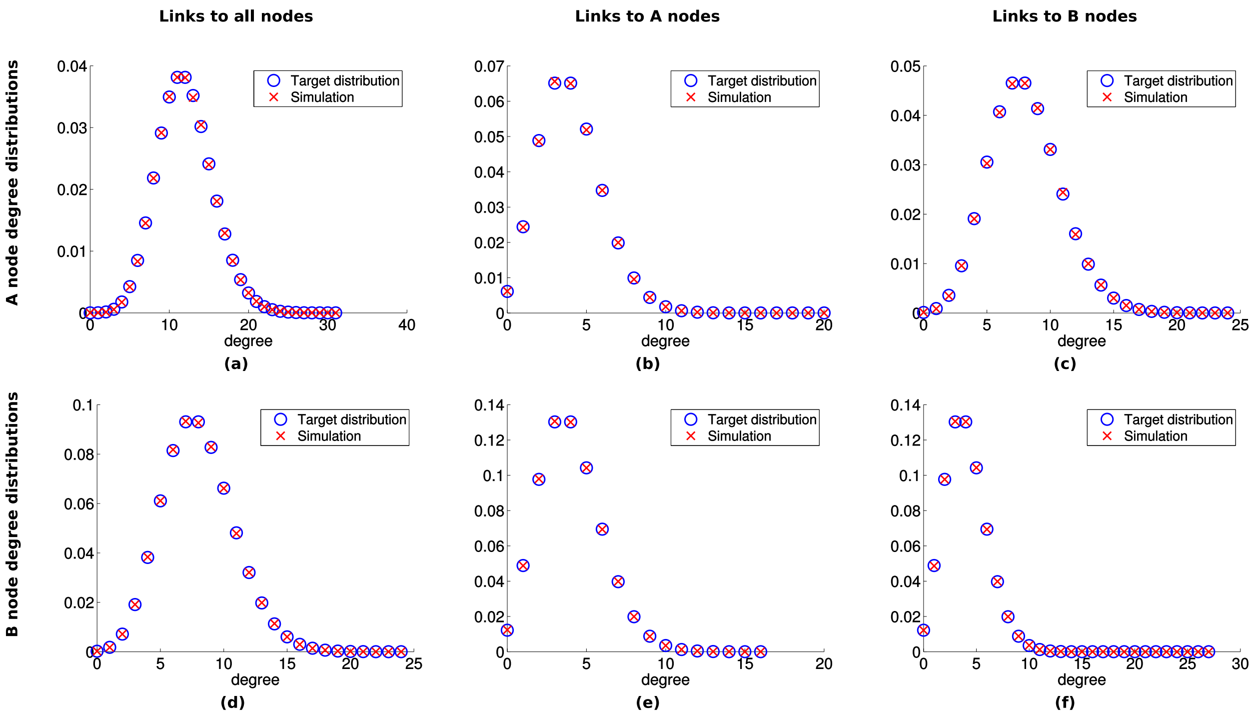

Fig. 4. Local rules generate target distributions in a two-state network; general case. Shown are target distributions (circles) compared to agent-based simulations (crosses) designed to self-organize to the target distribution by using the relations (13) and (14). Rates are as follows: $w_{\mathrm{AB-AA}} = 0.01, w_{\mathrm{AB-BB}} = 0.02, w_{\mathrm{AA-AB}} = 0.04, w_{\mathrm{BB-AB}} = 0.04, \alpha_{k,l} = 0.05, \beta_{k,l} = 0.025\exp(-4)(2)^l$. Top is the degree distribution of A-nodes where (a) is the total degree distribution (b) is the degree distribution to A-nodes only (c) is the degree distribution to B-nodes only. Bottom is the degree distribution of B-nodes where (d) is the total degree distribution (e) is the degree distribution to A-nodes only (f) is the degree distribution to B-nodes only. Note differences in vertical scales.

exponential function $\exp[a(x-1)+b(y-1)]$ and then compare coefficients of x and y. We find

$$\begin{aligned} p_{\mathrm{A-B}} - \frac{b}{a} p_{\mathrm{B-A}} &= 0 \\ w_{\mathrm{AB-AA}} - \frac{a}{b}\frac{w_{\mathrm{AA-AB}}}{2} &= 0 \\ w_{\mathrm{AB-BB}} - \frac{b}{a}\frac{w_{\mathrm{BB-AB}}}{2} &= 0. \end{aligned} \tag{12}$$

Thus there is a wide range of feasible choices of process rates to satisfy these conditions for any given target distribution, with parameters a and b. A comparison between target distributions $G^A = \exp[4(x + 2y - 3)]/3$ and $G^B = 2\exp[4(x + 2y - 3)]/3$ and agent-based simulations subject to the relations (12), are shown in Fig. 3. The simulation results are in good agreement with the target degree distributions.

The aggregators A, in our sensor network are less abundant than the data recorders, B. There are many recorders per aggregator and there is low connectivity between aggregators, but high connectivity between recorders. This is due to our choice of G^A and G^B having equal exponents, hence the connectivity between aggregators and recorders which we wanted to be high is the same as the connectivity between recorders.

It could be advantageous in certain applications for the recorders to be connected with lower mean, potentially leading to the deployment of sensors over a larger area. We thus define two new target generating functions $G^A(x,y) = c_1\exp[a_1(x-1)+a_2(y-1)]$ and $G^B(x,y) = c_2\exp[b_1(x-1)+b_2(y-1)]$, such that the proportion of aggregators (c_1) is less than the proportion of recorders (c_2) and the mean of sensors connected of the same type (a_1 and b_2) is small, while the number of recorders per aggregator is large. Again the parameters are subject to constraints of symmetry and total aggregator and recorder density, which imply $c_1a_2 = c_2b_1$ and $c_1 + c_2 = 1$ respectively.

In order to design such a system we must introduce new processes. As in Section 4, we can use the same methodology when we allow for processes that can depend on the degree of the node. We therefore allow the state-change processes to be degree dependent. A-nodes can change state at rates $\alpha_{k,l}$ and B-nodes at rates $\beta_{k,l}$. As before, we must introduce two new generating functions $S(x,y) = \sum_{k,l}\alpha_{k,l}A_{k,l}x^ky^l$ and $T(x,y) = \sum_{k,l}\beta_{k,l}B_{k,l}x^ky^l$ for the state-change processes. The steady-state generating function PDEs for a network subject to these processes are given in the online supplementary material in Appendix C.

Substituting the target generating functions G^A and G^B into the PDEs gives two equations in six unknowns. This shows that the system is still under-determined and we have the freedom to impose additional constraints to arrive at a solution. Hence here we solve for the rewiring processes and state change processes separately.

For the rewiring equations, we can cancel the generating functions and compare coefficients of x and y to get the following relations between the rewiring rates

$$\begin{aligned} w_{\mathrm{AB-AA}} - \frac{a_1}{a_2}\frac{w_{\mathrm{AA-AB}}}{2} &= 0, \\ w_{\mathrm{AB-BB}} - \frac{c_2 b_2}{c_1 a_2}\frac{w_{\mathrm{BB-AB}}}{2} &= 0. \end{aligned} \tag{13}$$

Next, solving for the state-chance processes gives the relation between the state change rates $\alpha_{k,l}$ and $\beta_{k,l}$

$$\alpha_{k,l} - \mathrm{e}^{a_1+a_2-b_1-b_2}\,\frac{c_2}{c_1}\left(\frac{b_1}{a_1}\right)^k\left(\frac{b_2}{a_2}\right)^l\beta_{k,l} = 0. \tag{14}$$

Comparisons between target distributions, $G^A = \exp[4(x+2y-3)]/3$ and $G^B = 2\exp[4(x+y-2)]/3$, and agent-based simulations subject to the relations (13) and (14) are shown in Fig. 4. Compared to Fig. 3, the connectivity between aggregators and between aggregators and recorders remains the same, but there are fewer recorder to recorder connections, as per the design criteria. The results from the agent-based simulation are in good agreement with the target degree distributions.

6 Conclusion

In this paper we have proposed a method for the design of rules that let a network self-organize into a target steady-state degree distribution. This is achieved by first modelling the network using a heterogeneous moment expansion. The infinite dimensional system of ODEs from the heterogeneous approximation can then be converted into first order PDEs using generating functions, where the number of PDEs will depend on the number of states in the system. By substituting the target steady-state degree distribution into the generating function PDEs we derive algebraic consistency conditions, from which it is possible to determine which processes on a network result in the target degree distribution.

There are a number of caveats to the method proposed here, which concern the convergence to the desired state, the validity of the approximation and the applicability in the real world. First, the method proposed here generates a set of rules under which the desired state is stationary. However, it does not guarantee that this state is locally dynamically stable or globally attractive. Local stability can be tested by local stability analysis of the generating function PDE linearized around the desired state. This is a well established procedure, and hence has not been used in this paper. For systems with degree-independent rules also the global attractivity should not present a problem as these rules lead to linear systems, which have only a single attractor. For non-linear degree-dependent processes, multiple attractors can exist, thus global attractivity is hard to guarantee. However, the example of Pyragas control [24], for instance, shows that methods which only guarantee the existence but not stability of a solution can be useful in practise. In the design of a system, such methods, including the one here can be used to quickly narrow down the space of possible solutions. Any solution that is then considered for implementation in the real world will certainly first be tested in simulations, where local and global stability can be examined.

A second concern is the mathematical validity of the approach. The proposed method is exact except for the active neighbourhood approximation. This approximation is known to provide a highly accurate approximation for stationary states of dynamical networks [15]. The approximation relies on the absence of long-ranged correlation in the network. Such correlations can arise during transients, which is of little concern for the method proposed here, and in certain systems close to bifurcations. As a general rule, detrimental correlations will be present, first, when the network fragments on a global scale (such as the fragmentation transition in the adaptive voter model [12], [25]), or, second, when processes in the network lead to an over-abundance of certain meso-scale motifs, that far exceeds statistical expectations. In these cases other approximations need to be used that take the respective correlations into account. These approximations could be higher order heterogeneous approximations such as the heterogeneous pair approximation [14], [26], or the motif approximation developed in [25]. All of these lead to infinite systems of ODEs that can be transformed, and lead to PDEs, along the lines demonstrated in this paper.

The final caveat concerns applicability. On the one hand, at present there do not seem to be many applications where the self-organization to a certain degree distribution is needed. On the other hand, there is an enormous potential for self-organizing systems, from swarm robotics and mobile sensor networks, via synthetic biology, to smart power grids and traffic systems. It is perhaps unlikely that the self-organised degree distribution demonstrated here, based on very simple local rules, will find real world applications. However, we believe that this demonstration provides a proof of concept on which future work can build. Perhaps the biggest potential for applications lies in the field of swarm robotics, where it could enable small robots with very limited processing and communication capabilities to robustly assemble and/or communicate in desirable global formations. Recent papers [27]–[29] have demonstrated that the dynamics of swarms can be understood using network models. Using this network-level description of swarms the method proposed here could enable the design of collective dynamics in swarms of robotic agents. We anticipate that this application will necessitate further extensions to the method to deal with physical constraints imposed by space. For instance, to cut a link an agent might need to move away from a partner, which may necessarily lead to the cutting or formation of other links. To incorporate such complications the explicit tracking of larger network motifs by an approximation other than the heterogeneous active-neighbourhood approximation will probably be necessary. Thus such extensions, already discussed above, might not only improve the accuracy of the method, but also provide a crucial step toward implementation in a wide range of applications.

Appendix A
Heterogeneous approximation for binary adaptive network

Here we give the ODEs describing the evolution of the binary adaptive network described in Section 5, subject to constant rewiring and constant state-change processes. We use a heterogeneous active-neighbourhood approximation to track the abundance of nodes $N_{k,l}$, where $N \in \{A, B\}$. For nodes $A_{k,l}$ we find

$$\begin{aligned}
\frac{\mathrm{d}A_{k,l}}{\mathrm{d}t} &= w_{\mathrm{AB-AA}}\Big[(l+1)A_{k-1,l+1} - lA_{k,l} \\
&\qquad + \frac{\sum_{k',l'} l'A_{k',l'}}{\sum_{k',l'} A_{k',l'}}\left(A_{k-1,l} - A_{k,l}\right)\Big] \\
&\quad + w_{\mathrm{AB-BB}}\left[(l+1)A_{k,l+1} - lA_{k,l}\right] \\
&\quad + \frac{w_{\mathrm{AA-AB}}}{2}\big[(k+1)A_{k+1,l-1} - kA_{k,l} \\
&\qquad + (k+1)A_{k+1,l} - kA_{k,l}\big] \\
&\quad + \frac{w_{\mathrm{BB-AB}}}{2}\left[\frac{\sum_{k',l'} l'B_{k',l'}}{\sum_{k',l'} A_{k',l'}}\left(A_{k,l-1} - A_{k,l}\right)\right] \\
&\quad + p_{\mathrm{A-B}}\left[\left((k+1)A_{k+1,l-1} - kA_{k,l}\right) - A_{k,l}\right] \\
&\quad + p_{\mathrm{B-A}}\left[\left((l+1)A_{k-1,l+1} - lA_{k,l}\right) + B_{k,l}\right],
\end{aligned} \tag{15}$$

and similarly for $B_{k,l}$ nodes

$$\begin{aligned}
\frac{\mathrm{d}B_{k,l}}{\mathrm{d}t} &= w_{\mathrm{AB-AA}}\left[(k+1)B_{k+1,l} - kB_{k,l}\right] \\
&\quad + w_{\mathrm{AB-BB}}\Big[(k+1)B_{k+1,l-1} - kB_{k,l} \\
&\qquad + \frac{\sum_{k',l'} k'B_{k',l'}}{\sum_{k',l'} B_{k',l'}}\left(B_{k,l-1} - B_{k,l}\right)\Big] \\
&\quad + \frac{w_{\mathrm{AA-AB}}}{2}\left[\frac{\sum_{k',l'} k'A_{k',l'}}{\sum_{k',l'} B_{k',l'}}\left(B_{k-1,l} - B_{k,l}\right)\right] \\
&\quad + \frac{w_{\mathrm{BB-AB}}}{2}\big[(l+1)B_{k-1,l+1} - lB_{k,l} \\
&\qquad + (l+1)B_{k,l+1} - lB_{k,l}\big] \\
&\quad + p_{\mathrm{A-B}}\left[\left((k+1)B_{k+1,l-1} - kB_{k,l}\right) + A_{k,l}\right] \\
&\quad + p_{\mathrm{B-A}}\left[\left((l+1)B_{k-1,l+1} - lB_{k,l}\right) - B_{k,l}\right]
\end{aligned} \tag{16}$$

Appendix B
Generating function PDEs for a binary adaptive network

We convert the infinite-dimensional systems of ODEs (15)-(16), into two coupled first-order PDEs by introducing two generating functions $G^A(x,y) = \sum_{k,l} A_{k,l}x^k y^l$ and $G^B(x,y) = \sum_{k,l} B_{k,l}x^k y^l$. We multiply (15) and (16) by x^k and y^l and sum over $k, l \geq 0$. In the steady state for G^A we find

$$\begin{aligned}
0 &= G^A_x\Big[(y-x)\left(\frac{w_{\mathrm{AA-AB}}}{2} + p_{\mathrm{A-B}}\right) \\
&\qquad + \frac{w_{\mathrm{AA-AB}}}{2}(1-x)\Big] \\
&\quad + G^A_y\left[(x-y)\left(w_{\mathrm{AB-AA}} + p_{\mathrm{B-A}}\right)\right. \\
&\qquad \left. + w_{\mathrm{AB-BB}}(1-y)\right] \\
&\quad + G^A\Big[w_{\mathrm{AB-AA}}\frac{\bar{G}^A_y}{\bar{G}^A}(x-1) \\
&\qquad + \frac{w_{\mathrm{BB-AB}}}{2}\frac{\bar{G}^B_y}{\bar{G}^A}(y-1) - p_{\mathrm{A-B}}\Big] \\
&\quad + p_{\mathrm{B-A}}G^B,
\end{aligned} \tag{17}$$

and similarly for $G^B(x,y)$

$$\begin{aligned}
0 &= G^B_x\left[(y-x)\left(w_{\mathrm{AB-BB}} + p_{\mathrm{A-B}}\right)\right. \\
&\qquad \left. + w_{\mathrm{AB-AA}}(1-x)\right] \\
&\quad + G^B_y\Big[(x-y)\left(\frac{w_{\mathrm{BB-AB}}}{2} + p_{\mathrm{B-A}}\right) \\
&\qquad + \frac{w_{\mathrm{BB-AB}}}{2}(1-y)\Big] \\
&\quad + G^B\Big[w_{\mathrm{AB-BB}}\frac{\bar{G}^B_x}{\bar{G}^B}(y-1) \\
&\qquad + \frac{w_{\mathrm{AA-AB}}}{2}\frac{\bar{G}^A_x}{\bar{G}^B}(x-1) - p_{\mathrm{B-A}}\Big]
\end{aligned} \tag{18}$$

$$+ p_{\mathrm{A-B}} G^A,$$

where, for example, $\bar{G}^A = G^A(1,1)$.

Appendix C
Generating function PDEs for a binary adaptive network with degree-dependent state-change rates

In Section 5 we introduce degree-dependent state change processes into the system, while rewiring remains constant. Hence A-nodes change state at a rate $\alpha_{k,l}$ and B-nodes change state at a rate $\beta_{k,l}$. We therefore introduce two new generating functions $S(x,y) = \sum_{k,l} \alpha_{k,l} A_{k,l} x^k y^l$ and $T(x,y) = \sum_{k,l} \beta_{k,l} B_{k,l} x^k y^l$. In the steady-state for G^A we find

$$0 = G_x^A \left[\frac{w_{AA-AB}}{2}(y - 2x + 1) \right] \quad \text{(19i)}$$

$$+ G_y^A \left[w_{AB-AA}(x-y) + w_{AB-BB}(1-y) \right] \quad \text{(19ii)}$$

$$+ G^A \left[w_{AB-AA} \frac{\bar{G}_y^A}{\bar{G}^A}(x-1) + \frac{w_{BB-AB}}{2} \frac{\bar{G}_y^B}{\bar{G}^A}(y-1) \right] \quad \text{(19iii)}$$

$$+ \frac{\bar{S}_x}{\bar{G}_x^A}(y-x) G_x^A + \frac{\bar{T}_x}{\bar{G}_y^A}(x-y) G_y^A - S + T, \quad \text{(19iv)}$$

and similarly for G^B

$$0 = G_x^B \left[w_{AB-BB}(y-x) + w_{AB-AA}(1-x) \right] \quad \text{(20i)}$$

$$+ G_y^B \left[\frac{w_{BB-AB}}{2}(x - 2y + 1) \right] \quad \text{(20ii)}$$

$$+ \left[w_{AB-BB} \frac{\bar{G}_x^B}{\bar{G}^B}(y-1) + \frac{w_{AA-AB}}{2} \frac{\bar{G}_x^A}{\bar{G}^B}(x-1) \right] G^B \quad \text{(20iii)}$$

$$+ \frac{\bar{S}_y}{\bar{G}_x^B}(y-x) G_x^B + \frac{\bar{T}_y}{\bar{G}_y^B}(x-y) G_y^B + S - T. \quad \text{(20iv)}$$

Substituting the target distributions $G^A(x,y)$ and $G^B(x,y)$ from section 5 into (19) and (20) gives two equations in six unknowns. This shows that the system is still under-determined and we have the freedom to impose additional constraints to arrive at a solution. Hence here we solve for the rewiring processes and state change processes separately. We set (19i) + (19ii) and (20i) + (20ii) equal to zero and solve to get a relation between the rewiring processes, while (19iv) and (20iv) give a relation between the state-change processes.

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