Topological stability criteria for synchronized coupled systems of non-identical oscillators

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Individual dynamical units that are coupled via an interaction network can synchronize spontaneously without central regulation. The propensity to synchronize depends on structural properties of the interaction network. Certain network properties such as heterogeneity and diameter have been studied extensively, however sometimes with conflicting conclusions. This indicates that beside global structural measures of the network also the local and mesoscale configurations of network nodes play a role that cannot be neglected. By constructing a graphical interpretation of Jacobi's signature criterion, we show that synchronization in phase oscillators can only be achieved if the network obeys necessary topological conditions. These analytical results thus identify instabilities that occur in subgraphs on the mesoscale. We then formulate and numerically confirm a conjecture that extends the stability conditions to large scale structures. Finally, we show that the proposed approach can be extended to an adaptive oscillator network, in which the coupling topology co-evolves with the dynamics of the coupled units.

Spontaneous synchronization of dynamically interacting units plays an important role in many different fields including biology, ecology, and engineering [1, 2]. The paradigmatic model proposed by Kuramoto [3] opened the field for detailed studies of the interplay between the structure of the interaction network and collective phenomena [4–7]. These studies have revealed the influence of various topological measures – such as the clustering coefficient, the diameter, and the degree or weight distribution – on the propensity to synchronize [8–10]. However, recent results [2, 11, 12] indicate that beside global topological measures also details of the exact local configuration can crucially affect synchronization. This highlights synchronization of phase oscillators as a promising example in which it may be possible to understand the interplay between local, global, and mesoscale constraints on stability, that severely limit the operation of complex technical and institutional systems [13, 14].

In this paper we apply Jacobi's signature criterion (JSC), to find necessary conditions for the stability of phase-locked solutions in undirected networks of non-identical phase oscillators. These conditions pertain to subgraphs that contain multiple nodes but are smaller than the entire network, thus imposing constraints on the mesoscale. By observing a regularity in the analytical conditions we then formulate a conjecture stating that for synchronization a spanning tree has to exist in which all interactions are reinforcing. Such conditions obtained from the JSC are complementary to statistical analysis of synchronization as they provide analytical insights pinpointing the sources of instabilities hindering synchronization.

The JSC (also known as Sylvester criterion) states that the number of negative eigenvalues of a matrix $\mathbf{J}$ equals the number of changes of sign in the sequence $1, D_1, \ldots, D_r$, where $D_q := \det(J_{ik}),\ i,k = 1,\ldots,q$, is the principal minor of order q and r is the rank of $\mathbf{J}$ [15]. In a stable system the sequence has to alternate in every step, i.e., the q-th order principal minor D_q has to have the sign of $(-1)^q$.

Stability analysis by means of JSC is well-known in control theory [15] and has been applied to problems of different fields from fluid- and thermodynamics to offshore engineering [16–18]. The applicability of JSC is presently limited mostly to systems with few degrees of freedom, as the criterion is formulated in terms of determinants. Numerical computation of determinants is typically accomplished by computing products of eigenvalues or singular values [19] and is thus more costly than numerical stability analysis of the system. The analytical evaluation of the criterion is impeded by the combinatorial growth of the number of terms. Dealing with this growth is the central difficulty in the present paper, which we address by proposing a convenient graphical notation.

STABILITY IN NETWORKS OF PHASE-OSCILLATOR

Following [20] we consider a system of N phase oscillators i whose time evolution is given by

$$\dot{x_i} = \omega_i + \sum_{j\neq i} A_{ij} \sin(x_j - x_i)\ , \quad \forall i \in 1 \ldots N\ . \tag{1}$$

Here, x_i, ω_i respectively, denote the phase and the intrinsic frequency of node i and $\mathbf{A} \in \mathbb{R}^{N\times N}$ is a weight matrix of an undirected, weighted interaction network. Two oscillators i, j are thus connected if $A_{ij} = Aji \neq 0$. The model can exhibit phase-locked states, which correspond to steady states of the governing system of equations. The local stability of such states is determined by the eigenvalues of the Jacobian matrix $\mathrm{J} \in \mathbb{R}^{\mathrm{N\times N}}$ defined by $J_{ik} = \partial\dot{x}_i/\partial x_k$. If all eigenvalues of J are negative then the state under consideration is asymptotically stable.

In the present system the Jacobian J is symmetric and thus admits analysis by the JSC. For obtaining a sufficient condition for stability we have to demand that the sign of the minors changes in every step of the sequence. This is impracticable for most larger systems because a) large determinants have to be evaluated and b) a large number of conditions have to be checked. However, because the sign must alternate in every step demanding $\mathrm{sgn}\,(D_q) = -1^q$ for some q already yields a necessary condition for stability. We note that this condition pertains to a specific subgraph of size q.

The stability condition that is found by considering a principal minor of given order q depends on the ordering of variables, i.e. the ordering of rows and columns in the Jacobian. The number of necessary conditions that is obtained can therefore be increased by considering different orderings [21]. By considering all possible reorderings one finds a condition constraining every subgraphs of size q that is present in the network. For distinguishing minors relying on different orderings of the variables, we define $S = \{s_1, \ldots, s_q\}$ as a set of q indices and $D_{q,S}$ as the determinant of the submatrix of $\mathbf{J}$, which is spanned by the variables $x_{s_1}, \ldots, x_{s_q}$. Therewith, the necessary conditions for stability read

$$\mathrm{sgn}\,(D_{q,S}) = (-1)^q, \quad \forall S,\ q = 1, \ldots, r. \tag{2}$$

GRAPHICAL NOTATION

Considering necessary conditions, avoids the difficulty (b), mentioned above, which leaves us to deal with difficulty (a) arising from the combinatorial explosion of terms that are needed to write out the conditions for increasing q. For instance in the common notation more than 700 terms are necessary for expressing the minors of order 6. Although we cannot circumvent this problem completely, progress can be made by employing a graphical notation that captures basic intuition and allows for expressing the principal minors more precisely.

For $q = 4$, $S = \{i, j, k, l\}$

$$\square := J_{ij}J_{jl}J_{lk}J_{ki} + J_{ij}J_{jk}J_{kl}J_{li} + J_{ik}J_{kj}J_{jl}J_{li}$$

$$|\cdot| := J_{ik}^2 J_{jl}^2 + J_{ij}^2 J_{kl}^2 + J_{il}^2 J_{jk}^2$$

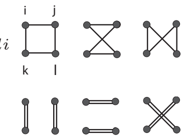

FIG. 1. Examples for the graphical notation. Symbols denote the sum over all non-equivalent possibilities to build the depicted subgraph with the q vertices $\in S$. Plotted are two example terms and their algebraic and topological equivalents.

For arriving at the graphical notation consider that the Leibniz formula for determinants [22] implies that (i) a minor $D_{q,S}$ is a sum over $q!$ elementary products $J_{i_1 j_1} \cdot \ldots \cdot J_{i_q j_q}$; and (ii) in each of these products every index $s_i \in S$ occurs exactly twice. Let us now interpret the Jacobian $\mathbf{J}$ as the weight matrix of an undirected, weighted graph $\mathcal{G}$. A Jacobian element J_{ij} then corresponds to the weight of an edge connecting vertices i and j. We can now relate products of the Jacobian elements to subgraphs of $\mathcal{G}$ spanned by the respective edges. For instance $J_{ij}J_{jk}$ is interpreted as path i-j-k, $J_{ij}J_{jk}J_{ki}$ as a closed path from i to j to k and back to i.

Based on the above, we associate every subgraph in $\mathcal{G}$ with the value found by multiplying the weights of the corresponding edges. This enables us to express the minors of $\mathbf{J}$ as sums over subgraphs. Because of property (i), each term of a minor $D_{q,S}$ corresponds to a subgraph with q edges. Because of property (ii), these subgraphs are composed of sets of cycles in $\mathcal{G}$: Every index $s_i \in S$ occurs either with multiplicity two on a diagonal element of $\mathbf{J}$, or, with multiplicity one, on two off-diagonal elements of $\mathbf{J}$. In the former case, the respective factor corresponds to a self-loop of $\mathcal{G}$, i.e., to a cycle of length $n = 1$; in the latter case, there is a set of factors J_{ij} $i \neq j$ corresponding to a closed path of edges, i.e., a cycle of length $n > 1$.

In summary, properties (i) and (ii) imply that all minors $D_{q,S}$ of a matrix $\mathbf{J}$ can be decomposed into cycles of $\mathcal{G}$. This sets the stage for the graphical notation which consists of a basis of symbols and a summation convention. The basis of symbols is given by $\times, |, \triangle, \square, \pentagon, \ldots$ denoting cycles of length $n = 1, 2, 3, 4, 5, \ldots$. The summation convention stipulates that in a minor $D_{q,S}$, every product of symbols denote the sum over all non-equivalent possibilities to build the depicted subgraph with the q vertices $s_1, \ldots, s_q$ (cf. Fig. 1).

With these conventions the first 6 principal minors can be written as

$$D_1 = \times \tag{3a}$$
$$D_2 = \times \cdot \times - | \tag{3b}$$
$$D_3 = \times \cdot \times \cdot \times - \times \cdot | + 2\triangle \tag{3c}$$
$$D_4 = \times \cdot \times \cdot \times \cdot \times - \times \cdot \times \cdot | + | \cdot | + 2 \times \cdot \triangle - 2\square \tag{3d}$$
$$D_5 = \times \cdot D_4 - 2\triangle \cdot | + 2⬠ \tag{3e}$$
$$D_6 = \times \cdot D_5 - | \cdot | \cdot | + 4\triangle \cdot \triangle + 2\square \cdot | - 2⬡ \tag{3f}$$

The Eqs. (3) allow us to infer the general formation rule for minors:

$$D_q = \sum \text{all combinations of symbols with} \sum n = q, \tag{4}$$

where symbols with $n > 2$ appear with a factor 2 that reflects the two possible orientations in which the corresponding subgraphs can be paced out. Symbols with an even (odd) number of edges carry a negative (positive) sign related to the sign of the respective index permutation in the Leibniz formula for determinants.

ZERO ROW SUM

Because of fundamental conservation laws many systems, including the standard Kuramoto model, have Jacobians with zero row sum, such that $J_{ii} = -\sum_{j \neq i} J_{ij}$. Using this relation we can remove all occurrences of elements J_{ii} from the Jacobian and its minors. In the topological reading this substitution changes the corresponding networks by replacing a self-loop at a vertex i by the negative sum over all edges of $\mathcal{G}$ that connect to i.

The simplification of the minors due to the zero row sum condition can be understood using the example of Fig. 2. Replacing the self-loops, the first term of every minor $D_{q,S}$, $\times^q$, turns out to be $(-1)^q$ times the sum over all subgraphs meeting the following conditions: First, the subgraph contains exactly q edges. Second, there is at least one edge connecting to every vertex $\in S$. Third, every edge connects to at least one vertex $\in S$. And fourth, no vertex $\notin S$ has more than one edge connected to it. By means of elementary combinatorics it can be verified that all other terms of $D_{q,S}$ cancel exactly those subgraphs in $\times^q$ that contain cycles. This enables us to express the minors in another way. Defining $\Phi_{q,S}$ as the sum over all acyclic subgraphs of $\mathcal{G}$ that contain q edges and all vertices $\in S$ we can write

$$D_{q,S} = (-1)^q \Phi_{q,S}. \tag{5}$$

We remark that Kirchhoff's Theorem [23], which has previously been used for the analysis of dynamical systems [24, 25], appears as the special case of Eq. (5) where $q = N - 1$.

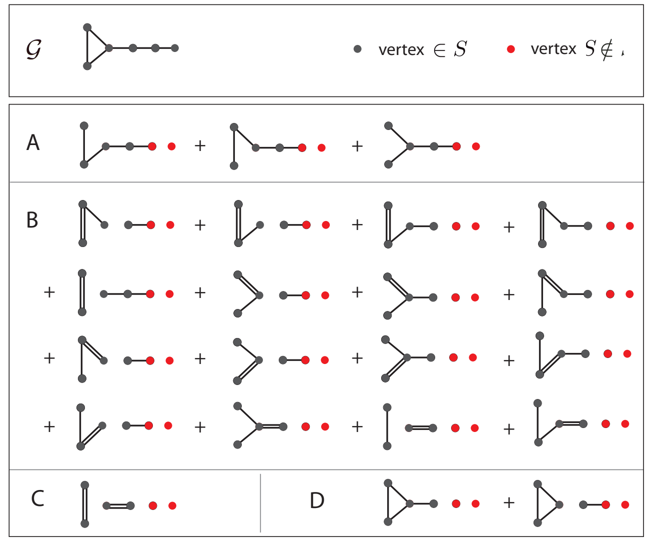

FIG. 2. Symbolic calculation of a determinant using the graphical notation. Consider the minor $D_{4,S}$ of the graph $\mathcal{G}$ sketched above. If S is chosen to be the set of vertices plotted in grey, the terms of Eq. (3d) can be written as $\times \cdot \times \cdot \times \cdot \times = A + B + C + 2D$, $-\times \cdot \times \cdot | = -(B + 2C)$, $| \cdot | = C$, $2 \times \cdot \triangle = -2D$ and $-2\square = 0$. It thus follows that $D_{4,S} \equiv \Phi_{4,S} = A$ is the the sum over all acyclic subgraphs of $\mathcal{G}$ that contain q edges and all vertices $\in S$.

TOPOLOGICAL STABILITY CONDITIONS

Let us shortly summarize what we obtained so far. The topological reading of determinants maps a symmetric, Jacobian $\mathbf{J}$ with zero row sum onto a graph $\mathcal{G}$, whose weight matrix is given by the off-diagonal part of $\mathbf{J}$. The minors of $\mathbf{J}$ can then be interpreted as sums over values associated with subgraphs of $\mathcal{G}$. Combining the Eqs. (2) and (5), the algebraic stability constraints on the minors of $\mathbf{J}$ translate into

$$\Phi_{q,S} > 0, \quad \forall S, \ q = 1, \ldots, r. \tag{6}$$

We emphasize that the graph $\mathcal{G}$ is not an abstract construction, but combines information about the physical interaction topology and the dynamical units. For example, if a graph $\mathcal{G}$ has disconnected components, there is a reordering of the variables x_i, such that $\mathbf{J}$ is block diagonal. This implies that the spectra of different topological components of $\mathcal{G}$ decouple and can thus be treated independently.

From Eq. (6) we can immediately read off a weak *sufficient* condition for stability: Because $\Phi_{q,S}$ is a sum over products of the J_{ij}, a Jacobian with $J_{ij} \geq 0 \ \forall i, j$ is a solution to Eq. (6) irrespective of the specific structure of $\mathcal{G}$ [26]. By contrast, if $J_{ij} < 0$ for some i, j, the existence of solutions of Eq. (6) is dependent on the topology. In the following we investigate which combinations of negative J_{ij} in a graph $\mathcal{G}$ lead to the violation of at least one of the Eqs. (6). For this purpose we first explore the restric-

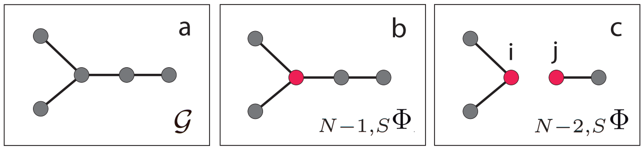

FIG. 3. Necessary stability condition for trees. (a,b) If $\mathcal{G}$ is a tree, $\Phi_{N-1,S}$ has only one term, $\mathcal{G}$ itself (vertex indices $\notin S$ are marked red). The condition $\Phi_{N-1,S} > 0$ then allows for an even number of negative edges in $\mathcal{G}$. If the number of negative edges is > 0 (e.g. 2), we can find a pair of vertices i, j connected by a negative edge and choose $S = \{k | k \neq i, j\}$ such that $\Phi_{N-2,S} = \Phi_{N-1,S}/J_{ij} < 0$ (c). It follows that if $\mathcal{G}$ is a tree, stability requires that all edges are positive.

tions that Eq. (6) imposes on simple topological building blocks. Thereafter, we piece the results for the different example topologies together to formulate a conjecture for stability conditions in large networks.

In Fig. 3 we illustrate, that if $\mathcal{G}$ a tree, $J_{ij} \geq 0 \; \forall i, j$ is not only a sufficient but also a necessary condition for stability. The reasoning in the figure translates one-to-one to any tree-like subgraph of a general graph $\mathcal{G}$. More precisely: If $\mathcal{G}$ possesses edge induced subgraphs, which are trees, stability requires that all edges belonging to these subgraphs correspond to positive entries of the Jacobian. In other words stability requires that edges corresponding to negative entries can only appear in cyclic parts of the network.

In Fig. 4, we summarize results for different cyclic example graphs. The figures show that stability restricts the maximum number of edges corresponding to negative elements, as well as their position, and their strength. In the example networks the maximum number of these *negative edges* that can be reconciled with stability is one less than the number of independent cycles. Further the negative edges must be placed such that the graph that is obtained by removing the negative edges is still connected.

The results from the examples considered so far can be summarized by saying that stability of the synchronized state requires that a spanning tree must exist in which every edge corresponds to a positive element of the Jacobian. We conjecture that this result holds also in the general case, so that stability in any network requires that the graph $\mathcal{G}$ must have a spanning tree of positive edges, i.e., if all negative edges are removed from $\mathcal{G}$, the remaining graph must still be connected.

In order to check the conjecture stated above numerically, we generated ensembles of 10^8 connected graphs of size $N = 25$ and fixed mean degree $\langle k \rangle$. In each graph, we assigned a negative weight α to all but $N - 1$ randomly chosen edges. The remaining edges were assigned weight 1. We then checked for each graph, whether the graph had a positive spanning tree and calculated the largest non-trivial eigenvalue λ of the corresponding Jacobian.

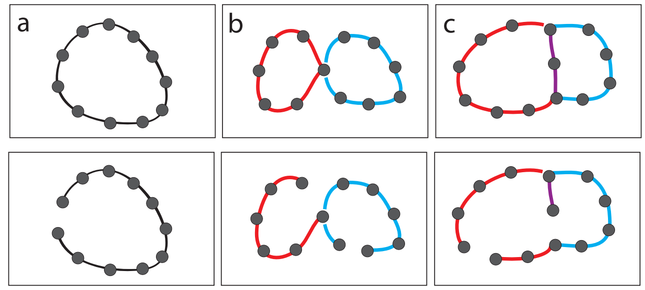

FIG. 4. Necessary stability conditions for cyclic example topologies. (a) If $\mathcal{G}$ is a cycle, it can have at most one negative edge whose strength $|\alpha|$ is bounded by a cycle length dependent relation (cf. Fig 5). (b) If $\mathcal{G}$ is composed of two cycles that share one vertex, each of these cycles can have at most one negative edge of bounded strength. (c) If $\mathcal{G}$ is composed of two cycles that share a sequence of edges, every unbranched sequence of edges (red, violet, blue), can at most have one negative edge, while the total number of negative edges may not exceed 2. Removing the maximum number of negative links from the example topologies reveals that the permissible graph is still connected (lower panels).

Let C be a cycle of length N, which includes $N - 1$ positive edges $c_1, \ldots, c_{N-1}$ and one negative edge α. Then stability requires that

$$|\alpha| < \frac{c_1 \cdot c_2 \cdot \ldots \cdot c_{N-1}}{\sum \text{all distinct products of } (N-2) \text{ factors } c_i}, \quad (7)$$

i.e. for N=3, $|\alpha| < \frac{c_1 c_2}{c_1 + c_2}$, for N=4, $|\alpha| < \frac{c_1 c_2 c_3}{c_1 c_2 + c_1 c_3 + c_2 c_3}$ and so forth.

FIG. 5. Stability sets a cycle-length dependent upper bound on the strength of a negative edge in a cycle. Note, that inserting an additional positive edge c_N in an cycle of length N decreases the upper bound on $|\alpha|$ irrespective of the value of c_N. In this sense, one can say that the longer the cycle the more restrictive the stability condition.

The procedure was repeated for different values of α.

Among the 10^9 generated test graphs, more than 98% did not contain a positive spanning tree. Of these networks none were found to be stable which corroborates the conjecture. Among the graphs that did contain a positive spanning tree (ca. 10^7), stability depended on the specific topology and the value of $|\alpha|$ (cf. Fig. 6). As expected from Fig. 5, the fraction of networks that are stable although they obey the necessary condition decreases with increasing $|\alpha|$.

ADAPTIVE COUPLING

Above we assumed that the system under consideration is described by a Jacobian that is symmetric and has zero row sums. We now consider an example where the zero row sum criterion is only met in certain net-

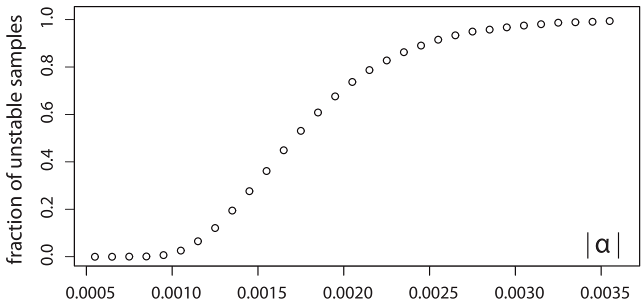

FIG. 6. Numerical test of the hypothesized stability condition. The fraction of matrices $\mathbf{J}$ that have a positive largest eigenvalue although the corresponding graph $\mathcal{G}$ possesses a positive spanning tree is plotted against $|\alpha|$. The continuous transition from 0 to 1 confirms that the upper bound on $|\alpha|$ depends on the exact position of the negative edge.

works. For this purpose we turn to adaptive networks (AN), in which the topology of the network coevolves with the dynamics of oscillators [27–29].

We consider a system of N phase oscillators that evolve according to Eq. (1), while the coupling strength A_{ij} evolves according to

$$\frac{\mathrm{d}}{\mathrm{dt}} A_{ij} = \cos(x_j - x_i) - b \cdot A_{ij}. \tag{8}$$

The first term in Eq. (8) states that the more similar the phases of two vertices the stronger reinforced is their connection, the second term guarantees convergence. In a stationary, phase-locked state state $A_{ij} = \cos(x_j - x_i)/b$ and all oscillators oscillate with a common frequency $\Omega = \frac{1}{N}\sum_i \omega_i$. The stability of this state is governed by a symmetric Jacobian. Defining $o_{ij} := \frac{1}{b}\cos^2(x_j - x_i)$, $m_i := -\sum_{j\neq i} o_{ij}$, $s_{ji} := \sin(x_j - x_i)$ it can be written in the form

$$\mathbf{J} = \left(\begin{array}{ccc|ccc} -b & 0 & 0 & s_{21} & s_{12} & 0 \\ 0 & -b & 0 & s_{31} & 0 & s_{13} \\ 0 & 0 & -b & 0 & s_{32} & s_{23} \\ \hline s_{21} & s_{31} & 0 & m_1 & o_{12} & o_{13} \\ s_{12} & 0 & s_{32} & o_{12} & m_2 & o_{23} \\ 0 & s_{13} & s_{23} & o_{13} & o_{23} & m_3 \end{array}\right) \tag{9}$$

which is shown here for $N = 3$. The marked partitioning separates two blocks on the diagonal. The upper one is a diagonal submatrix of size $L \times L$, $L := N(N-1)/2$, the lower one is a $N \times N$ symmetric submatrix with zero row sum, which we denote as $\mathbf{j}$.

Let us start our analysis by focusing on the upper left block of $\mathbf{J}$. In the chosen ordering of variables the first L minors D_q satisfy the stability condition Eq. (2) iff $b > 0$. Concerning the minors of order $q > L$, the following conventions prove advantageous: We define $D_{L+n,S}$ as the determinant of the submatrix of $\mathbf{J}$, which is spanned by all variables A_{ij} and the n variables x_{s_i}. Further, $D_{0+n,S}$ denotes the determinant of the submatrix of $\mathbf{j}$, which is solely spanned by the n variables x_{s_i}.

We find that

$$D_{L+n,S} = (-1)^L b^{L-n} \cdot F(D_{0+n,S}), \tag{10}$$

where F is the linear mapping $F : o_{ij} \to \cos(2(x_j - x_i))$. As the submatrix $\mathbf{j}$ is symmetric and has a zero row sum, its minors, $D_{0+n,S}$, can be rewritten using Eq. (5)

$$D_{L+n,S} = (-1)^{L+n} b^{L-n} F(\Phi_{n,S}), \tag{11}$$

where $\Phi_{n,S}$ refers to subgraphs of the graph $\mathcal{G}$ defined by the off-diagonal entries of $\mathbf{j}$. Stability requires that $\operatorname{sgn}(D_{L+n,S}) = \operatorname{sgn}\left((-1)^{L+n}\right)$. As the necessary stability condition $b > 0$ determines b^{L-n} to be positive, it follows that in a stable system

$$F(\Phi_{n,S}) > 0, \quad \forall S, n = 1 \ldots r. \tag{12}$$

Comparison with Eq. 6 reveals that the stability conditions are related by the mapping F. The implications of this relationship between the stability of adaptive and non-adaptive networks will be discussed in detail in a seperate publication.

CONCLUSIONS

In the present paper we analyzed necessary conditions for local asymptotic stability of stationary and phase-locked states in networks of phase oscillators. Using a graphical interpretation of Jacobi's signature criterion we first formulated conditions for the stability of small subgraphs and then generalized these in a conjecture stating that stability requires the existence of a spanning tree in which every edge corresponds to a positive element of the Jacobian matrix.

Our results provide an analytical angle that is complementary to statistical analysis of network synchronizability. Where statistical approaches reveal global features impinging on the propensity to synchronize, our approach can pinpoint specific defects precluding synchronization. We note that such defects can occur all scales, corresponding to the violation of the signature criterion in subgraphs of different size. This highlights synchronization of phase oscillators as a simple but intriguing example in which instabilities can arise from local, global or mesoscale structures. In the future the approach proposed here may provide a basis for further investigation of these instabilities.

In real-world systems testing the conditions identified here requires information on the stationary phase profile. This limits the applicability of our approach for synchronizing systems which do not synchronize naturally. However, we note that even in such systems it may be possible to stabilize an existing unstable phase-locked state, e.g. by delayed-feedback control [30]. Based on the observed phase profile in the stabilized state one can then the identified conditions to search for structures that preclude synchronization when the controller is turned off. A more direct application of the present results is possible in networks with designed phase profiles [31]. Given a coupling topology and a desired phase profile it is often relatively easy to find a set of natural frequencies for which the phase profile is stationary, but not necessarily stable. Here the necessary stability conditions provide constrains on the stable profiles that may be realized in a given coupling topology.

Finally, the present results demonstrate the applicability of Jacobi's signature criterion to large networks. In principle the criterion can be applied to all systems in which the Jacobian is a Hermitian matrix. In the present paper we additionally assumed that the Jacobian has zero row sums. An example of the application of Jacobi's signature criterion in a large system where the zero row sum condition is violated is presented in [21]. However, in this work the authors considered only the first few minors of the Jacobian because of the combinatorial growth of expressions. We hope that future authors will find the graphical notation proposed here useful for mitigating this difficulty.

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