

# Computing 3D Bifurcation Diagrams

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## Abstract

The nature and localization of critical parameter sets called bifurcations is a central issue in nonlinear dynamical system theory. Codimension-1 bifurcations that can be observed due to one parameter variations form surfaces in three dimensional parameter spaces. Some bifurcations of higher codimension can be identified as intersections of these surfaces and bare insights about the global dynamics of the system. Here we describe an algorithm that combines adaptive triangulation with the approach of generalized models and theory of complex systems to compute and visualize such bifurcations in a very efficient way.

The method enables us to gain extensive insights in the local and global dynamics not only in one special system but in whole classes of systems. To illustrate these capabilities we present an example of a generalized eco-epidemic model.

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Bifurcations are sudden qualitative transitions in the long term dynamics of systems due to parameter variations. The localization of these critical parameter values is often a central task in the analysis of systems corresponding to many areas of science. Famous examples are the formation of Rayleigh-Bénard convection cells in hydrodynamics Swinney and Busse (1981), the onset of Belousov-Zhabotinsky oscillations in chemistry Agladze and Krinsky (1982); Zaikin and Zhabotinsky (1970) or the breakdown of the thermohaline ocean circulation in climate dynamics Titz et al. (2002); Dijkstra (2005).

Applied research is mainly restricted to codimension-1 bifurcations, which can be directly observed in experiments Guckenheimer and Holmes (1983). To observe bifurcations of higher codimension at least two parameters have to be set to its critical values. Therefore these bifurcations are rarely seen in experiments. Nevertheless the detection of higher-codimension lokal bifurcations bares informations which are also of interest from an applied point of view, like instance the presence of global bifurcations which are otherwise hard to detect or the presence of quasiperiodic or chaotic parameter regions

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Kuznetsov (1995). In recent years the approach of so-called generalized models facilitated the localization of these higher-codimension bifurcations in applied modeling. Generalized models are ODE model systems where the right hand sides are not necessarily restricted to specific functional forms Gross et al. (2004); Gross and Feudel (2006). Thus, each generalized model represents a class of conventional models and each bifurcation of higher codimension found in a generalized model may also play a role in many conventional models of the same class. For this reason the approach of generalized model helps on the one hand to transform the knowledge from theoretical bifurcation analysis into applied modeling. On the other hand it provides a lot of examples from applications to fundamental research.

Generalized models describe the local dynamics close to steady states. Although the steady states can not be computed directly a normalization procedure allows for a local stability analysis of a steady state Gross and Feudel (2006). Using classical algebraic methods combined with modern computer algebra systems implicit conditions for steady state bifurcations can be found. Concerning codimension-1 bifurcations these conditions describe hypersurfaces in parameter space on which the bifurcation points are located. Intersections between such hypersurfaces can be identified as higher-codimension bifurcations. Therefore an efficient visualization tool for implicit functions is needed locate these bifurcations in parameterspace.

In Stiefs et al. (2008) we demonstrated how the approach of generalized models, computer algebra assisted furcation analysis and a modern adapted algorithm for curvature dependent triangulation form a powerful tool in order to compute and visualize these bifurcations in parameterspace. In the following we explain the method in a short before we give an example of an eco-epidemic model.

Many real world systems can be described by a low dimensional set of variables  $X_1, \dots, X_N$  where the time evolution is given by a set of ordinary differential equations

$$\dot{X}_i = F_i(X_1, \dots, X_N, p_1, \dots, p_M), i = 1 \dots N \quad (1)$$

. The functions  $F_i(X_1, \dots, X_N, p_1, \dots, p_M)$  are in general nonlinear functions consisting of several terms. Since we use the approach of generalized models we do not have to specify the exact functional form of these terms. Let us consider a generalized system where each variable  $X_i$  is related to one gain term  $G_i(X_1, \dots, X_N)$  and two loss terms  $L_{1i}(X_1, \dots, X_N)$  and  $L_{2i}(X_1, \dots, X_N)$ .

$$\dot{X}_i = F_i = G_i(X_1, \dots, X_N) - L_{1i}(X_1, \dots, X_N) - L_{2i}(X_1, \dots, X_N) \quad (2)$$

Since we aim to analyze the local stability of steady states we assume that such a steady state exists  $X^* = X_1^*, \dots, X_N^*$ , which is true for many systems. Since we can not compute  $X^*$  directly without specific functions we need to normalize our system. Following Gross and Feudel (2006) we substitute normalized

variables  $x_i = X_i/X_i^*$  and normalized terms  $g_i(x) = G_i(X_1^*x_1, \dots, X_N^*x_N)/G_i(X^*)$  as well as the normalized loss terms  $l_{j_i}(x) = L_{j_i}(X_1^*x_1, \dots, X_N^*x_N)/L_{j_i}(X^*)$ , with  $j = 1, 2$  and get the normalized generalized model

$$\dot{x}_i = \alpha_i(g_i(x) - m_{x_i}l_{1_i}(x) - \tilde{m}_{x_i}l_{2_i}(x)). \quad (3)$$

The new parameters  $\alpha_i$  can be identified as the characteristic timescale of each variable and  $m_{x_i}$  and  $\tilde{m}_{x_i} = 1 - m_{x_i}$  are weight factors between the two loss terms of  $x_i$ . The more gain and loss terms are involved, the more weight factors appear which describe the influence of each term relative to the others at the steady state. In many cases these parameters can be estimated from theoretical reasoning or from experimental data. Due to the normalization the values of the variables and generalized gain and loss terms in the steady state are now well known, namely equal to one by definition.

The local stability of the steady state is characterized by the eigenvalues of the Jacobian matrix in the steady state. The steady state becomes unstable when the real part of at least one eigenvalue becomes positive. Thus, two bifurcation types are of interest: bifurcations of saddle-node type where a real eigenvalue becomes positive or Hopf bifurcations where a pair of complex conjugated eigenvalues crosses the imaginary axis. To obtain implicit functions for these bifurcation conditions we use the determinant of the Jacobian for the saddle-node type bifurcations and the method of resultants for the Hopf bifurcations as test functions Guckenheimer et al. (1997).

In the Jacobian the derivatives of the normalized gain and loss terms with respect to the normalized variables in the steady state appear as additional unknown parameters. These are the so-called generalized parameters which describe the required information on the mathematical form of the generalized terms. However, these parameters have in general a well defined meaning in the context of the application Gross and Feudel (2006); Stiefs et al. (2008).

The implicit functions derived from the Jacobian are testfunctions for the bifurcation surfaces. To visualize these surfaces we approximate them by a sets of triangles. The triangulation algorithm we use is a simplification of the algorithm proposed in Karkanis and Stewart (2001) but extended in order to approximate also borders and non-smooth regions.

We find bifurcation points by computing the root of our testfunctions, by a Newton-Raphson method (e.g. Kelley (2003)). Once we have found 3 points as vertices of a first triangle we can find adjacent triangles by computing points outside of this so-called seed triangle. The exact procedure is given in Stiefs et al. (2008). The main point is to adapt the size of the adjacent triangles to the local curvature of the surface. Therefore we use the first additional point outside the seed triangle to construct an interim triangle to approximate the local curvature of the surface. Then we increase the distance of the additional

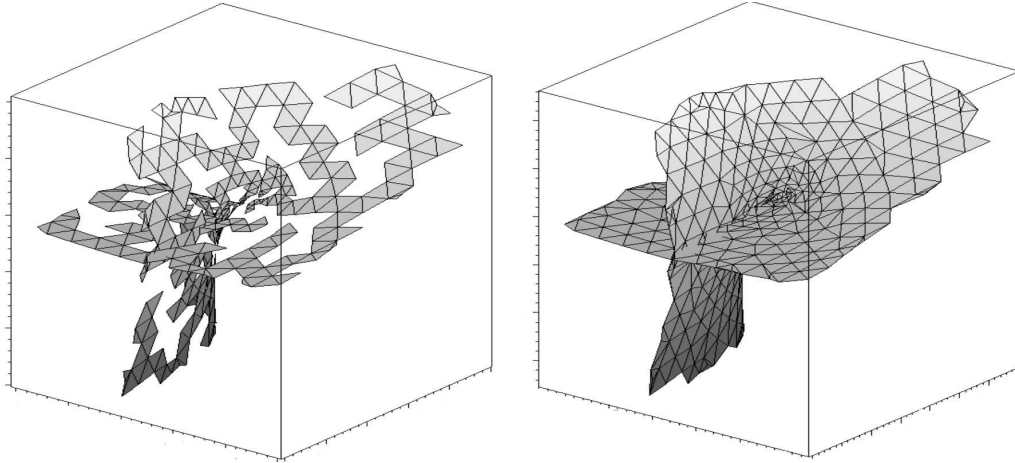


Fig. 1. Triangulation of a self intersecting surface

point to the seed triangle if the curvature is low or decrease the distance in the opposite case. In this way we can extend the seed triangle and each following triangle by other triangles with adequate size. This process is called the growing phase.

An overlapping of triangles is not allowed if the angle of the intersection is low. Consequently we end up with a mesh of triangles covering the whole surface in the prescribed parameter region. The left hand side of Figure shows such a mesh for a bifurcation surface of interesting shape. We will discuss the implications of this bifurcation surface later in our example.

After the growing phase the space between the triangles of the mesh becomes filled up with triangles as well. In this filling phase the vertices at both sides of the gaps become connected to build new triangles covering the gap. If these triangles are too large according to the local surface curvature we compute new points in between. Then we split the large triangle into smaller triangles by connecting the vertices with the new points.

When the filling phase is completed the whole bifurcation surface covered by triangles as seen at the right hand of Fig. . Now we prepare the surface for the final bifurcation diagram given in Fig. . Instead of displaying the edges of the triangles we project certain values from the axis on the surface like level lines in a map. This enhances the imagination of the viewer for the three-dimensional structure and enables to relate certain points at the surface to specific parameter values. Before we discuss the bifurcation diagram more in detail we give a short introduction into the eco-epidemic model behind it.

We consider a predator-prey model Eqs.(4) where a disease spreads upon the predator. We distinguish between susceptible, infected and recovered predators denoted by  $Y_S$ ,  $Y_I$  and  $Y_R$  respectively. The latter may be immune for some time before they become susceptible again.

$$\begin{aligned}
\dot{X} &= SX(1 - \frac{X}{K}) - G(X)(Y_S + Y_R + \alpha Y_I), \\
\dot{Y}_S &= EG(X)(Y_S + Y_R + \alpha Y_I) + \delta Y_R - M_S Y_S - \lambda(Y_S, Y_I), \\
\dot{Y}_I &= \lambda(Y_S, Y_I) - (M_S + \mu)Y_I - \gamma Y_I, \\
\dot{Y}_R &= \gamma Y_I - \delta Y_R - M_S Y_R
\end{aligned} \tag{4}$$

The prey  $X$  is assumed to grow logistically with a specific growth rate  $S$  and a capacity  $K$ . The amount of consumed prey per healthy predator is described by the generalized functional response  $G(X)$  while the amount per infected predator may be reduced by a factor  $\alpha$ . The consumed biomass is converted into biomass of healthy predators with an efficiency  $E$ . This efficiency may again be reduced by a factor  $\beta$  for the infected predators. The infection rate is given by the generalized incidence function  $\lambda(Y_S, Y_I)$ . All other processes are assumed to be linear. The parameter  $\delta$  is the rate for the loss of immunity and  $\gamma$  the rate of recovery. An additional mortality due to the infection is given by the parameter  $\mu$ .

After the renormalization we obtain the timescales, weight factors and the generalized parameters as bifurcation parameters. To discuss each parameter in detail is beyond the scope of this introduction to the method. Here we name only the parameters of the example diagram and discuss more general implications of bifurcations. In Fig. we see a bifurcation diagram of the normalized model with the generalized parameters

$$\begin{aligned}
g_x &:= \left. \frac{\partial g(x)}{\partial x} \right|_{x^*}, \\
l_i &:= \left. \frac{\partial l(y_s, y_i)}{\partial y_i} \right|_{y_s^*, y_i^*}
\end{aligned} \tag{5}$$

and the weight factor  $m_s$  of the natural mortality term of the susceptible predators as bifurcation parameters.

We see a Hopf bifurcation surface (red) and a surface of bifurcations of saddle-node type (transparent blue). The Hopf bifurcation surface is exactly the surface we used as an example surface above. The intersection line of both surfaces is a line of codimension-2 Gavrilov-Guckenheimer bifurcations where a zero eigenvalue in addition to a purely imaginary pair of complex conjugate eigenvalues can be found (Gavrilov, 1980; Guckenheimer, 1981). It is known that from a Gavrilov-Guckenheimer bifurcation a homoclinic bifurcation and a Neimark-Sacker bifurcation emerge. The first is related to sudden population bursts while the latter is related to the emergence of quasiperiodic motion. Due to these additional bifurcations in the neighborhood of the Gavrilov-Guckenheimer bifurcation, parameter regions of quasiperiodic dynamics exist and chaos is likely to occur.

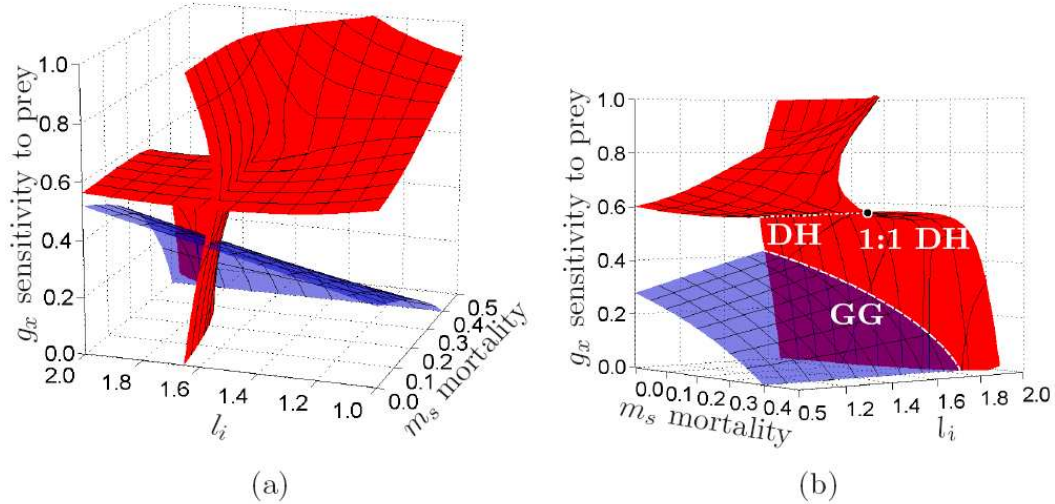


Fig. 2. Bifurcation diagram of the generalized eco-epidemic model. A surface of Hopf bifurcations (red) and a surface of saddle-node type bifurcations (transparent blue) are shown. The bifurcation parameters are the generalized parameters  $g_x$  and  $l_i$  which are strongly related to the functional form of the underlying processes  $g(x)$  and  $l(y_s, y_i)$  respectively and  $m_s$  describing the influence of natural mortality on the susceptible predators.

The topological form of the Hopf bifurcation surface which intersects itself is called Whitney umbrella. Examples of Whitney umbrella type bifurcations in applied models are very rare. One was found by Gross (2004) also published in (Stiefs et al., 2008). This Hopf bifurcation surface is twisted around a codimension-3 1:1 resonant double Hopf point characterized by two identical pairs of complex conjugated eigenvalues. As Fig. shows, a line of codimension-2 double-Hopf bifurcations emerges from this point. At this line where the Hopf bifurcation surface intersects itself two pairs of purely imaginary complex conjugated eigenvalues can be found. It has been shown that chaotic parameter regions generally exist close to a double Hopf bifurcation (Kuznetsov, 1995).

This example shows how the proposed method can be used to gain insights about the global dynamics of a class of system by analyzing local stability properties. These insights of the generalized model can be used to find the related dynamics in conventional models.

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